Aims :

1. Effects of flows on the magnetic field in high conductive plasma
2. Anti-dynamo theorem
3. Dynamo scenarios
   - Solar type dynamo
   - α^2 dynamo
   - cross-helicity dynamo?

Action of the plasma flow on the magnetic flux

Magnetic flux is frozen in plasma flow if diffusion term is neglected. Let $\Phi = \int \mathbf{B} \cdot \mathrm{d}\mathbf{S}$. The change of the flux is due to change of the induction $\frac{\partial \mathbf{B}}{\partial t}$ and due to the moving boundary $\delta t \mathbf{u} \times \mathrm{d}\mathbf{l} \cdot \mathbf{B} = -\delta t \mathbf{u} \times \mathbf{B} \cdot \mathrm{d}\mathbf{l}$. Thus the flux rate change is

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} - \oint_C \mathbf{u} \times \mathbf{B} \cdot \mathrm{d}\mathbf{l} = 0$$

or

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \int \left(\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{u} \times \mathbf{B}) \right) \cdot \mathrm{d}\mathbf{S} = 0$$

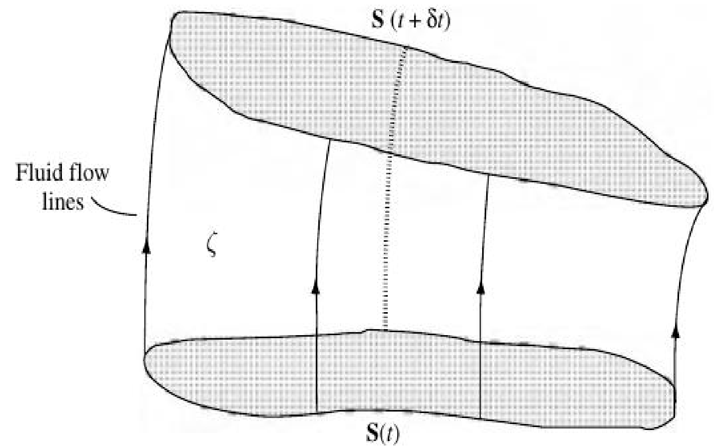


Figure 1.

Using vector calculus identity the induction equation can be rewritten as follows:

$$\frac{\partial \mathbf{B}}{\partial t} = \underbrace{(\mathbf{B} \cdot \nabla)\mathbf{u}}_{\text{stretching}} \; \underbrace{-\mathbf{B}(\nabla \cdot \mathbf{u})}_{\text{compression}} \; \underbrace{-(\mathbf{u} \cdot \nabla)\mathbf{B}}_{\text{advection}} \; \underbrace{+\eta \Delta \mathbf{B}}_{\text{diffusion}}$$

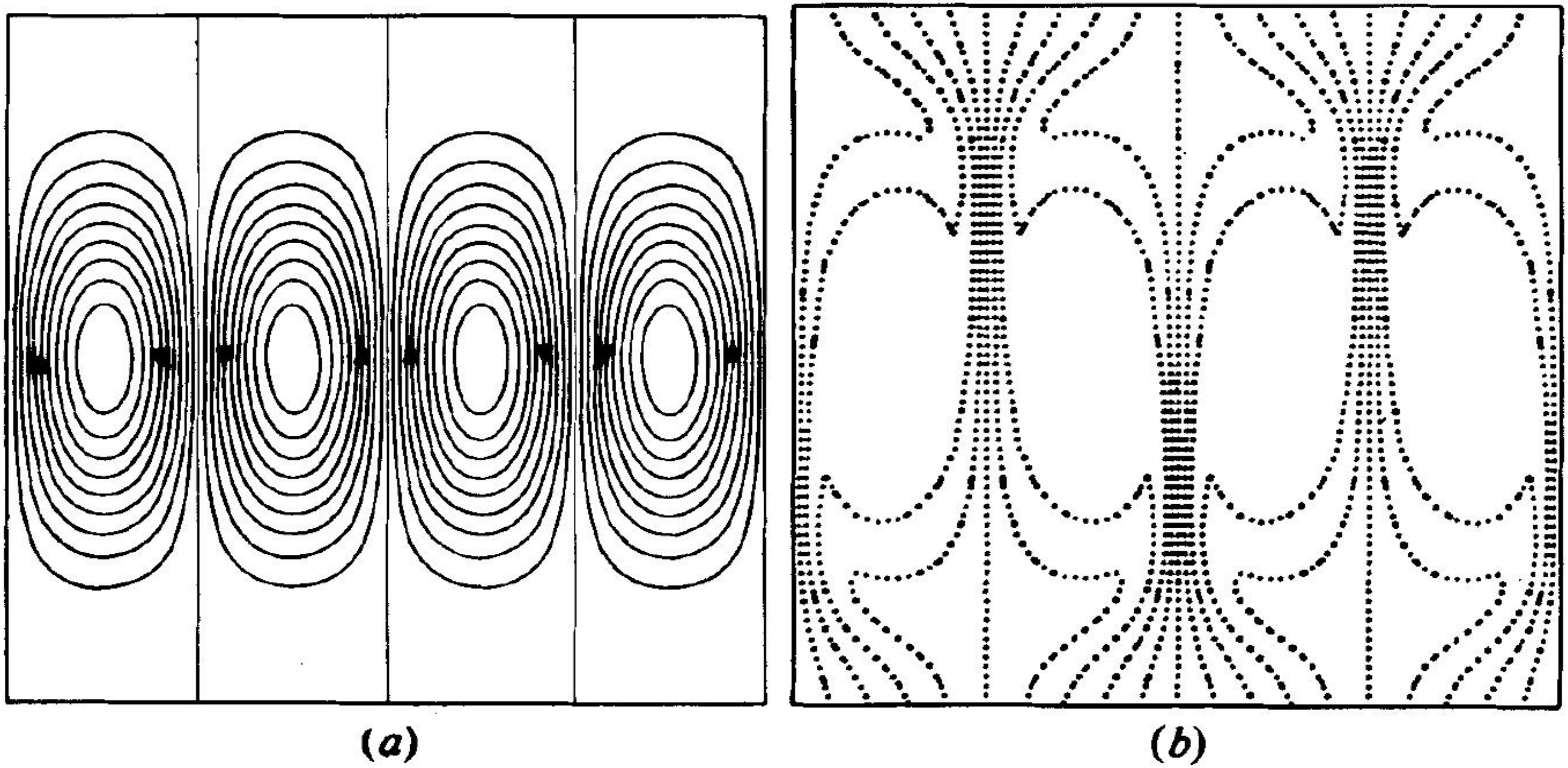
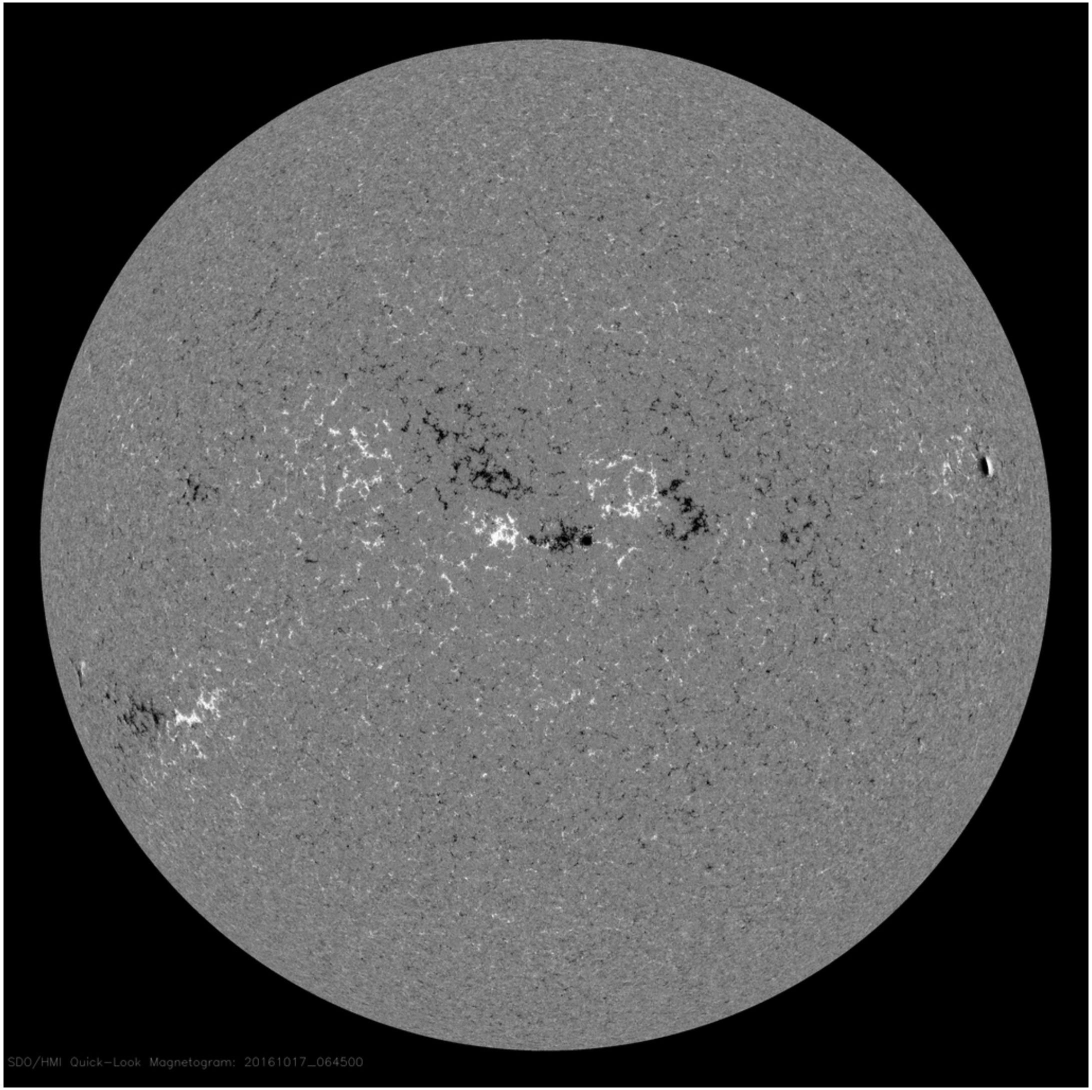


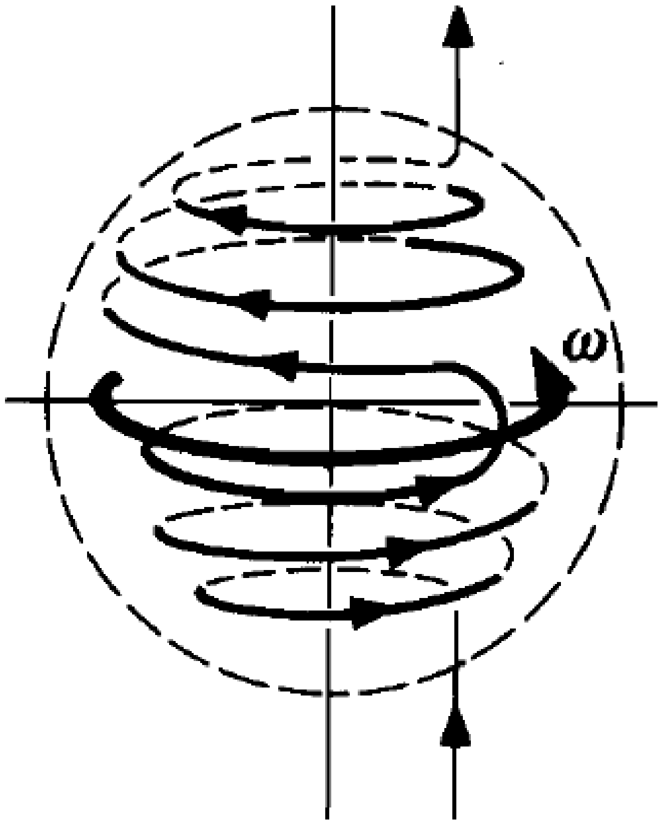
Fig. 3.5 Concentration of flux into ropes by a convective layer ($R_m = 10^3$); (a) streamlines ψ = cst. where ψ is given by (3.85); (b) lines of force of the resulting steady magnetic field. (From Weiss, 1966.)

Compression and advection, SDO/HMI, 17/10/2016



Stretching

Consider the cylindrical coordinate system: r, ϕ, z, axisymmetric rotational flow: $(0, r\Omega(r,z), 0)$, then the toroidal field is winding from poloidal by DR:



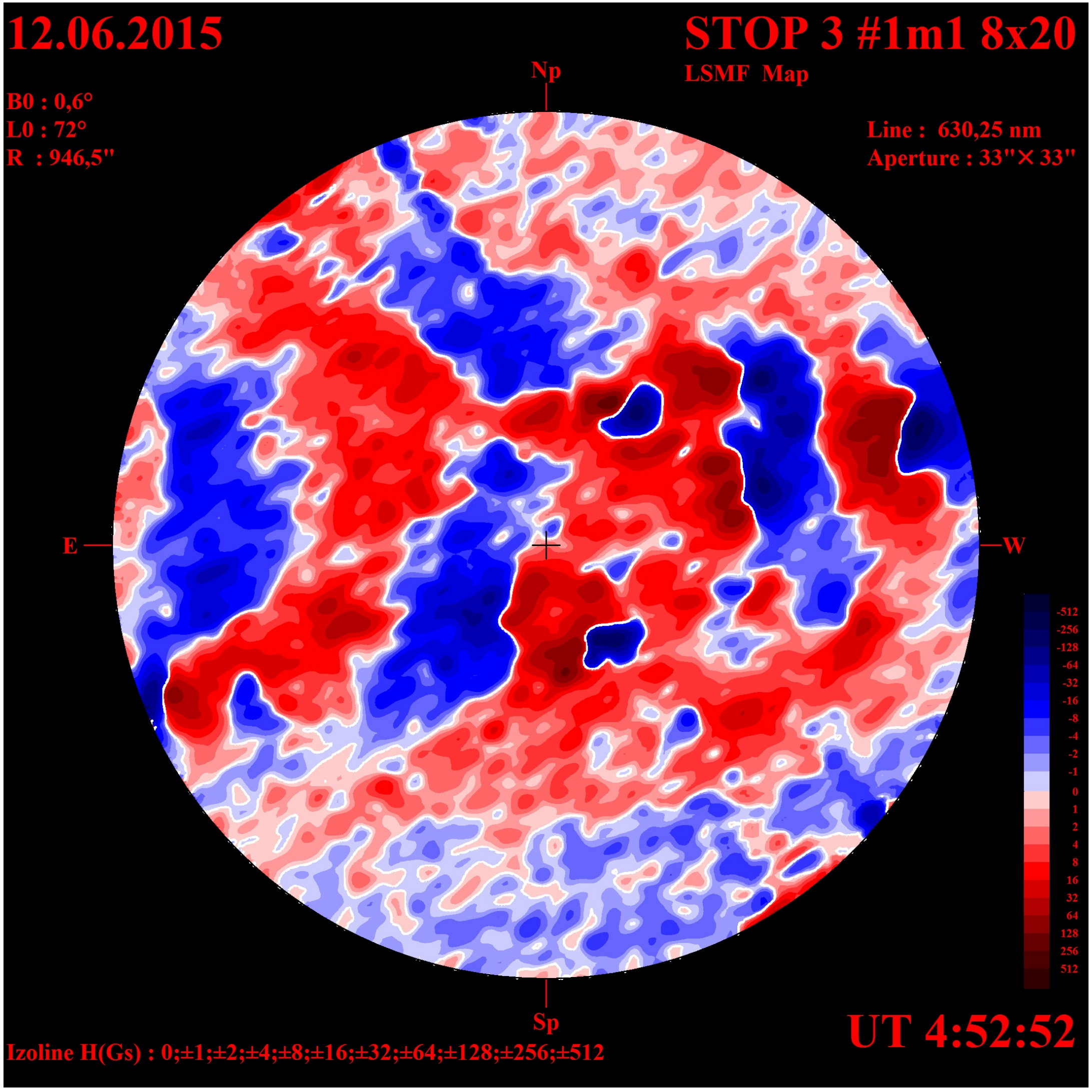
$$\frac{\partial B_\phi}{\partial t} = r(\mathbf{B}_p \cdot \nabla)\Omega, \qquad (1)$$

$$\frac{\partial \mathbf{B}_p}{\partial t} = 0$$

$$\mathbf{B}_p = e_r B_r + e_z B_z$$

The steady state is known as the *Ferraro* law: $(\mathbf{B}_p \cdot \nabla)\Omega$. The production of the net flux of the toroidal field in Eq.(12): $\frac{\mathrm{d}}{\mathrm{d}t}\int B_\phi \mathrm{d}S_m = 0$!

Large-scale unipolar field stretched by DR, courtesy SSO



Suppose that we allow for both poloidal and azimuthal velocity and magnetic fields, which may be steady or unsteady, but which are axisymmetric:

$$\boldsymbol{B} = \boldsymbol{B^p} + \boldsymbol{B}^\phi, \boldsymbol{U} = \boldsymbol{U}^p + \boldsymbol{U}^\phi$$

Then the induction equation can be divided fo the poloidal and toroidal parts:

$$\begin{aligned}\partial_t \boldsymbol{B}^\phi &= \boldsymbol{\nabla} \times (\boldsymbol{U}^\phi \times \boldsymbol{B^p} + \boldsymbol{U^p} \times \boldsymbol{B^\phi}) + \eta \Delta \boldsymbol{B^\phi}, \\ \partial_t \boldsymbol{B}^p &= \boldsymbol{\nabla} \times (\boldsymbol{U}^p \times \boldsymbol{B^p}) - \boldsymbol{\nabla} \times \eta \boldsymbol{\nabla} \times \boldsymbol{B^p}\end{aligned}$$

Using cylindrical coordinates and $\boldsymbol{B^p} = \boldsymbol{\nabla} \times r^{-1} A \boldsymbol{e^\varphi}$, and $\boldsymbol{U}^\varphi = \boldsymbol{e}^\varphi r \Omega$:

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{B}^\phi}{\mathrm{dt}} &= r(\boldsymbol{B}^p \cdot \boldsymbol{\nabla})\Omega + \eta \nabla^2 \boldsymbol{B}^\varphi, \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \eta \nabla_*^2 A, \nabla_*^2 = \frac{\partial^2}{\partial z^2} + r \frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}\end{aligned}$$

Note that

$$\int_V A \frac{\mathrm{d}A}{\mathrm{d}t} \mathrm{d}V = \int_V A \eta \nabla_*^2 A \mathrm{d}V = -\eta \int_V (\boldsymbol{\nabla} A)^2 \mathrm{d}V < 0,$$

Therefore, A decays with time and B^{φ} will decay eventually as well:

$$\frac{\mathrm{d}\boldsymbol{B}^{\phi}}{\mathrm{dt}} = r(\boldsymbol{B}^{p}\cdot\boldsymbol{\nabla})\mathrm{sin}\theta\Omega + \eta\nabla^{2}\boldsymbol{B}^{\varphi}$$

Cowling's neutral point argument.

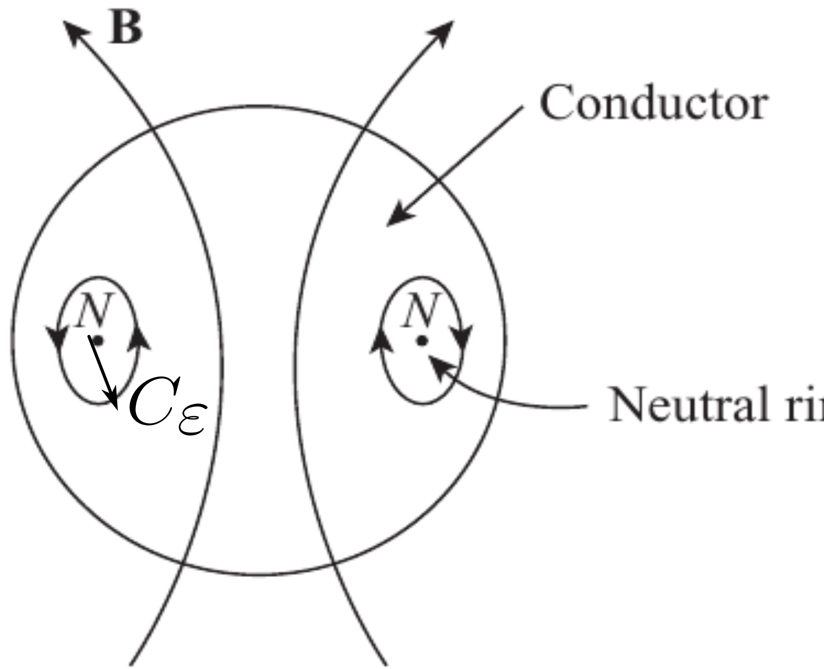
Suppose that we seek a steady, axisymmetric dynamo in which $\boldsymbol{B}^{p}$ is poloidal, **J** is azimuthal, and $\boldsymbol{U^{p}}$ is also poloidal, $\partial_t\boldsymbol{B}^{p}=0$, in vicinity C_{ε} it should be:

$$\int(\boldsymbol{U}^{p}\times\boldsymbol{B^{p}})\mathrm{dS}_{\varepsilon} = \eta\oint_{\boldsymbol{C\varepsilon}}\boldsymbol{B^{p}}\mathbf{d}\boldsymbol{r}$$

The right part scales as $\varepsilon\eta|B^{p}_{\varepsilon}|$, and the left part scales as $\varepsilon^{2}|U^{p}_{\varepsilon}||B^{p}_{\varepsilon}|$, decrease faster. so,

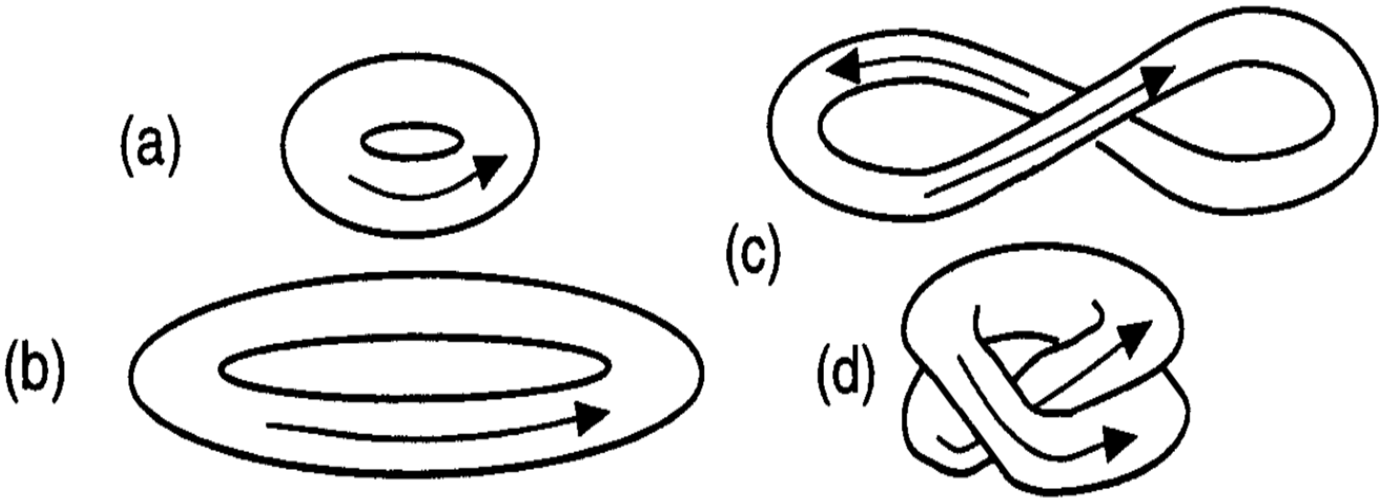
$$\int(\boldsymbol{U}^{p}\times\boldsymbol{B^{p}})\mathrm{dS} < \eta\oint\boldsymbol{B^{p}}\mathbf{d}\boldsymbol{r}$$

Induction effect of $\boldsymbol{U^{p}}\times\boldsymbol{B^{p}}$ does not compensate diffusion.



Dynamo scenarios

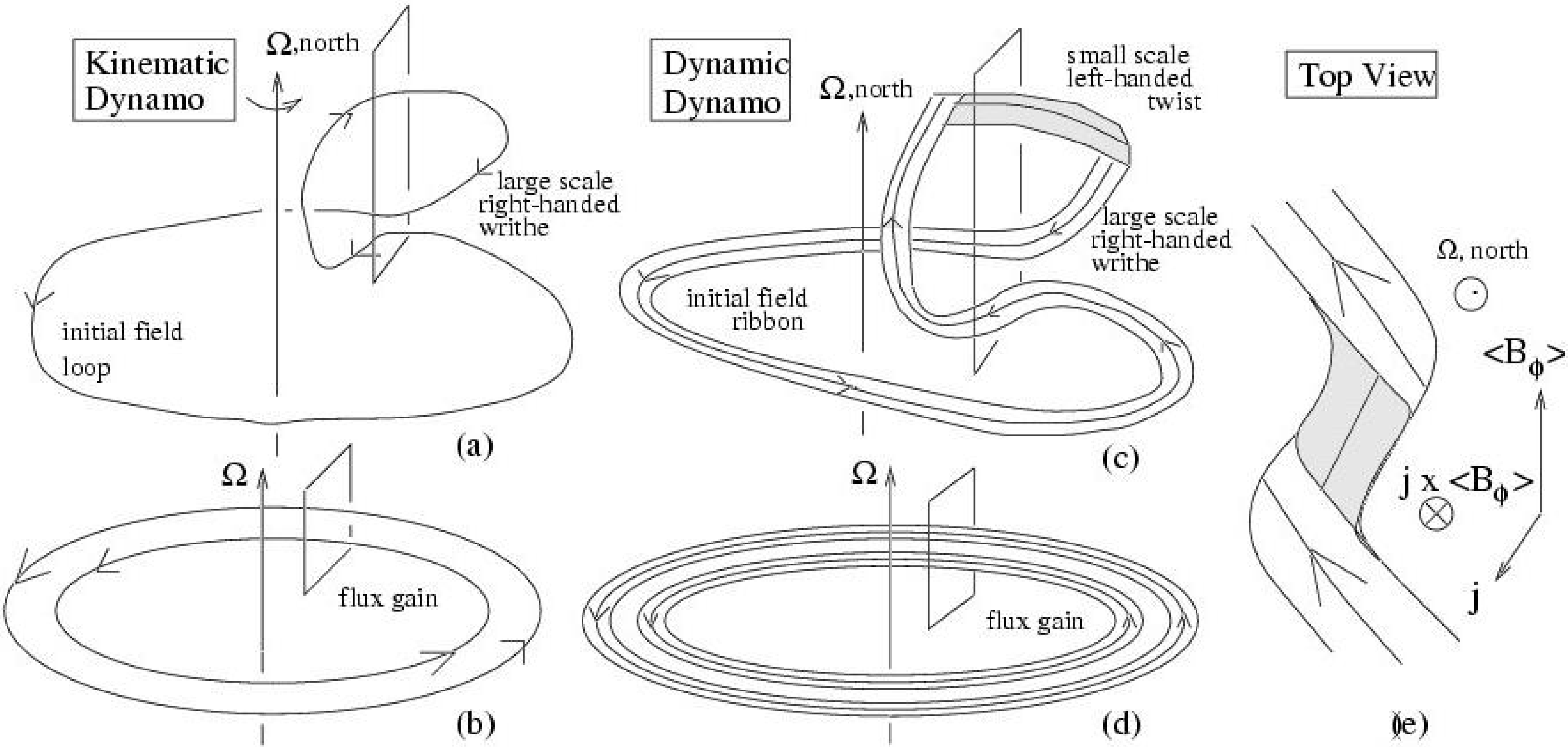
Stretch-twist fold dynamo Zeldovich 1966



- Magnetic field is frozen-in a high conductive turbulent fluid.
- Breaking reflection symmetry of turbulent motions.
  - This can result in spontaneous self-induction of magnetic field

Reconnection and diffusion are important part of dynamo instability

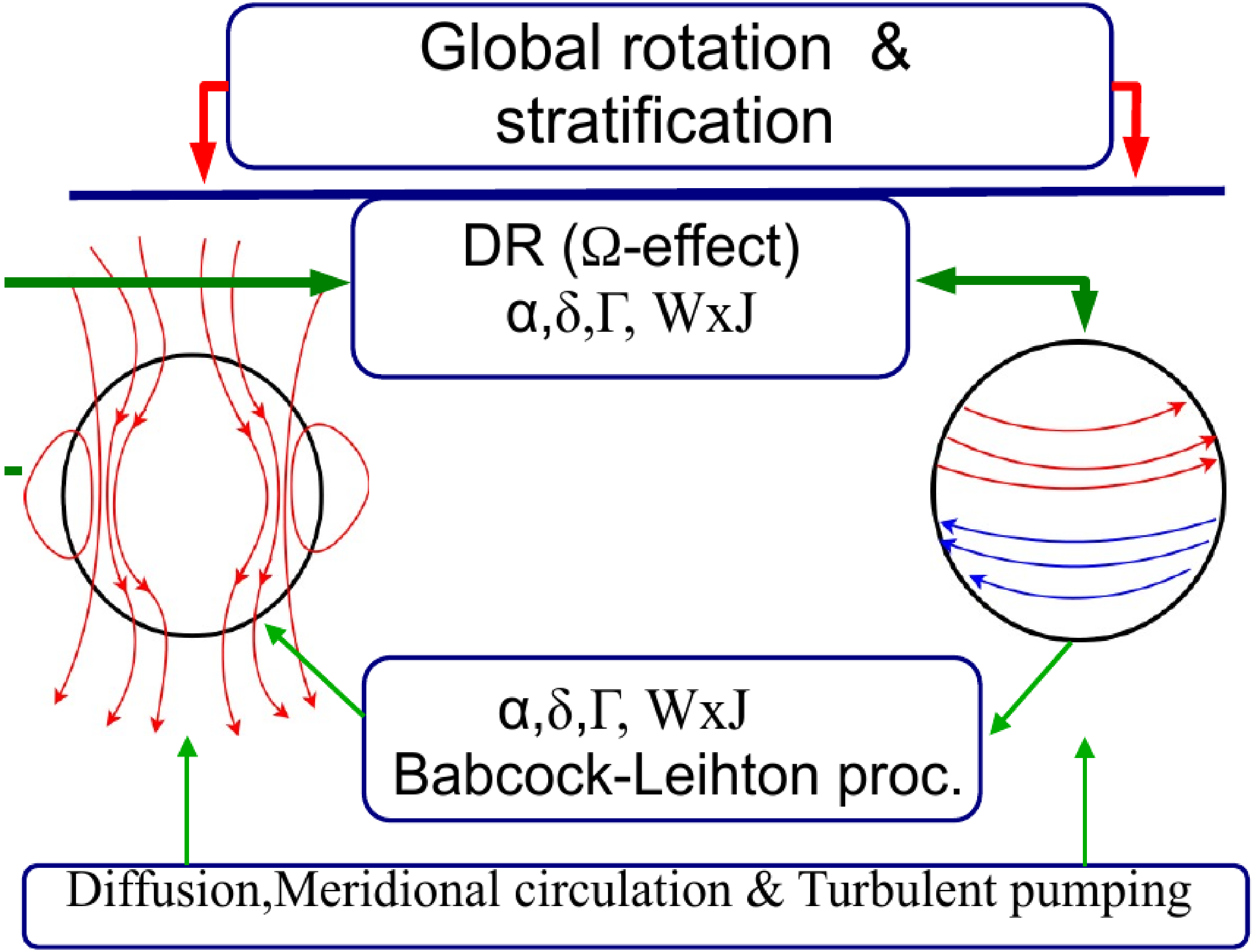
Solar case



Linear stage

Nonlinear stage

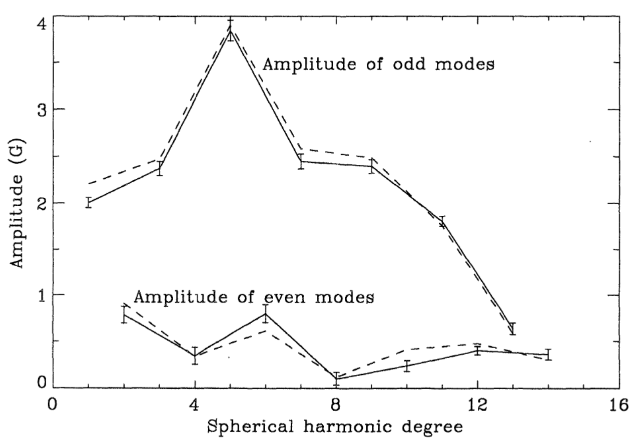
Solar dynamo scenarios

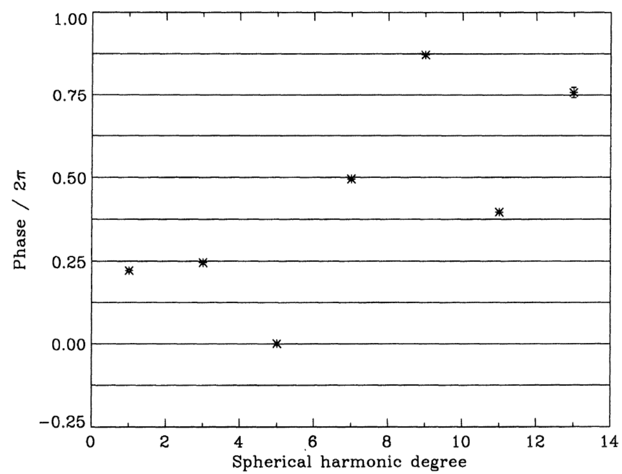


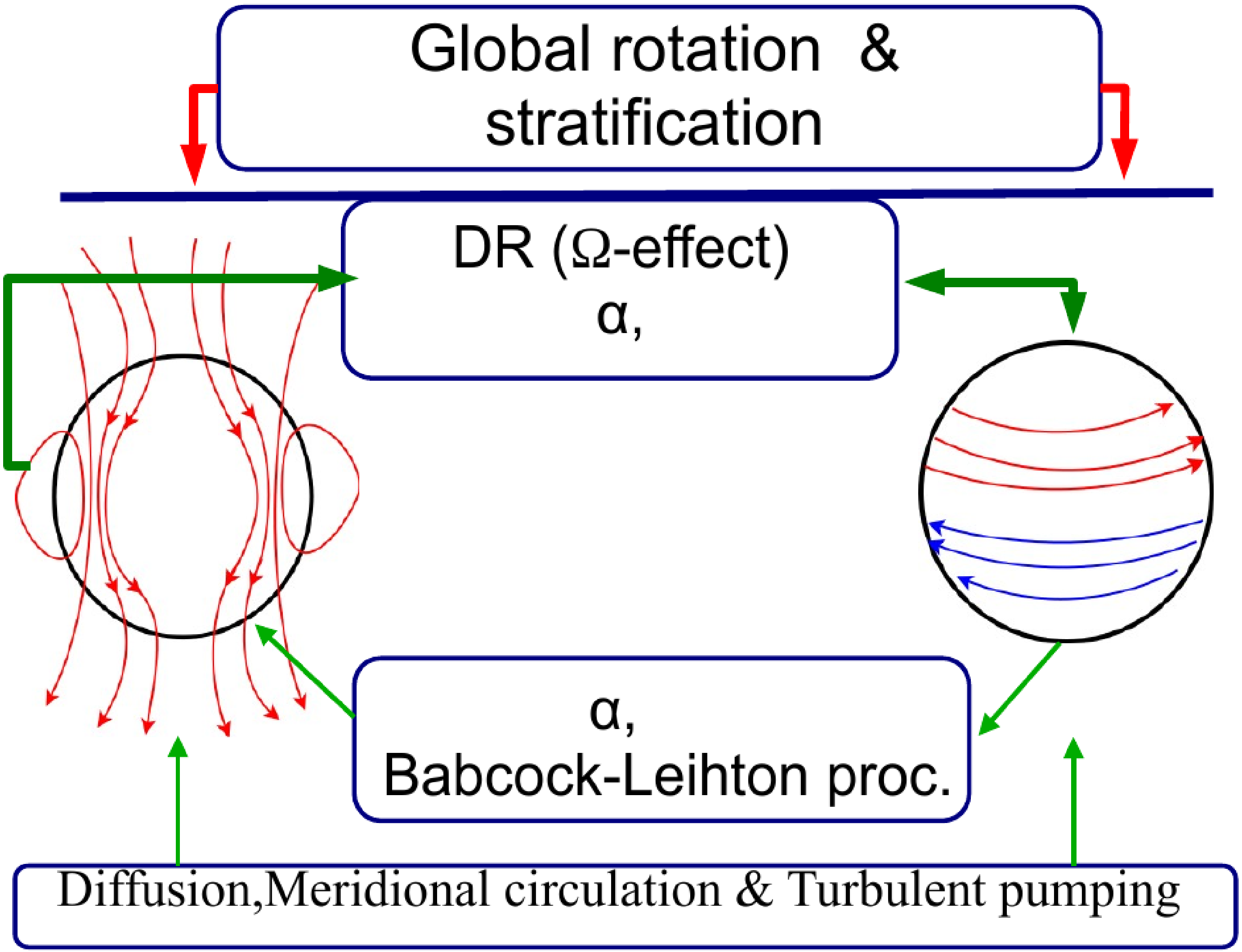
Observational restrictions on the solar dynamo scenarios

- $|\boldsymbol{B^p}| \ll |\boldsymbol{B^t}|$, where strength of the polar magnetic field during solar minims quantifies $\boldsymbol{B^p}$ and $\boldsymbol{B^t}$ is estimated by the magnetic field of the bipolar active regions
- Dynamo period and surface time-latitude diagrams
- Phase relations between polar and equatorial $\boldsymbol{B^p}$ and $\boldsymbol{B^t}$ also in space distributions, for example , $B_r B_\varphi < 0$ in the normal BMR for the magnetic activity maximum
- Extended 20 yr mode in Br (Stenflo 1994) and in the torsional oscillations (Ulrich 1998)
- Axisymmetric spherical harmonics spectrum (and phase of spherical harmonics)

$\langle B_r \rangle$






Global rotation &
stratification
DR (Ω-effect)
α,
α,
Babcock-Leihton proc.
Diffusion,Meridional circulation & Turbulent pumping

General properties of the dynamo solution

Let us consider the isotropic α effect and turbulent diffusion, and the shear is unidorm:

$$\begin{aligned}\partial_t\langle \boldsymbol{B}\rangle &= \boldsymbol{\nabla}\times(\boldsymbol{\mathcal{E}}+\langle \boldsymbol{U}\rangle\times\langle \boldsymbol{B}\rangle),\\ \boldsymbol{\mathcal{E}}=\langle \mathbf{u}\times\mathbf{b}\rangle &= \alpha\langle \mathbf{B}\rangle-\eta_T\nabla\times\langle \mathbf{B}\rangle,\end{aligned}$$

$$\langle \boldsymbol{B}\rangle=\boldsymbol{e}^{\varphi}B+\nabla\times\boldsymbol{e}^{\varphi}A,\langle \boldsymbol{U}\rangle=\boldsymbol{e}^{\varphi}\boldsymbol{(r\cdot S)},(\boldsymbol{e}^{\varphi}\cdot\boldsymbol{S})=0$$

After substitution:

$$\begin{aligned}\frac{\partial \boldsymbol{e}^{\varphi}B}{\partial t}= &\quad (\boldsymbol{B}^{p}\cdot\boldsymbol{\nabla})(\boldsymbol{e}^{\varphi}(\boldsymbol{r}\cdot\boldsymbol{S}))-\eta_T\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times B\,\boldsymbol{e}^{\varphi}\\ \frac{\partial \boldsymbol{e}^{\varphi}A}{\partial t} &= \alpha\boldsymbol{e}^{\varphi}B-\eta_T\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times A\,\boldsymbol{e}^{\varphi}\end{aligned}$$

Use the eigen-wave representation: $B,A\sim\exp(\sigma t+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})$ and $\boldsymbol{k}\cdot\boldsymbol{e}^{\varphi}=0$. Note, that

$$\frac{\partial \boldsymbol{e}^{\varphi}A}{\partial t}\rightarrow\sigma\boldsymbol{e}^{\varphi}A;\nabla\times\boldsymbol{e}^{\varphi}A\rightarrow\mathrm{i}\boldsymbol{k}\times\boldsymbol{e}^{\boldsymbol{\varphi}};\nabla\times\nabla\times A\,\boldsymbol{e}^{\varphi}\rightarrow-\eta_T\boldsymbol{k}^2A\,\boldsymbol{e}^{\varphi};$$

Therefore

$$\begin{aligned}\sigma B &= -\mathrm{i}(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi}A-\eta_T\boldsymbol{k}^2B\\ \sigma A &= \alpha B-\eta_T\boldsymbol{k}^2A\end{aligned}$$

Then compatibility condition determine the eigen values:

$$\begin{vmatrix} \sigma+\eta_T k^2 & \mathrm{i}(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi} \\ -\alpha & \sigma+\eta_T k^2 \end{vmatrix} = 0$$

The growing modes have $\mathrm{Re}(\sigma) > 0$. The solution of the determinant is

$$(\sigma+\eta_T k^2)^2 = -\mathrm{i}\alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi}$$

There are two roots: depending on the sign of $\alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi}$:

$$\sigma = -\eta_T k^2 + \begin{cases} \dfrac{1+\mathrm{i}}{\sqrt{2}}\sqrt{(\alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi})}, \mathrm{if} \quad \alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi} > 0 \\ \dfrac{1-\mathrm{i}}{\sqrt{2}}\sqrt{(-\alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi})}, \mathrm{if} \quad \alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi} < 0 \end{cases}$$

Suppose that $\sigma = \gamma - i\omega$, then, the dynamo frequency is $\omega = \pm\sqrt{\frac{1}{2}|\alpha(\boldsymbol{k}\times\boldsymbol{S})\cdot\boldsymbol{e}^{\varphi}|}$ and the normalized increment is

$$\hat{\gamma} = \frac{\gamma}{\eta_T k^2} = -1 + \sqrt{\mathrm{D}\sin\psi},$$

where, $D = \dfrac{\alpha S}{\eta_T^2 k^3}$ is the dynamo number, and ψ is angle between $\boldsymbol{k}$ and the shear vector $\boldsymbol{S}$.

Note, that, $S \sim \dfrac{\partial r \Omega}{\partial r} \sim \Omega$,

Wave propagation. Note, that $\boldsymbol{k}$ show propagation of phase of the dynamo wave. Therefore, the increment $\hat{\gamma}$ has maximum for $\psi = \pi/2$, i.e, the maximum of the dynamo efficiency is attained in direction which is perpendicular to the shear, in other words along the isorotation surface. **This is the Parker-Yoshimura rule.**

Wave frequency (Cycle period).

a)Linear growth. Consider the wavelength with maximum growth rate, i.e., consider wave with extreme of $\gamma = -\eta_T k^2 + \sqrt{\dfrac{\alpha k \Omega}{2}}$, (Brandenburg et al 2017)

$$\begin{aligned} \frac{\partial \gamma}{\partial k} &= -2\eta_T k + \frac{1}{2}\sqrt{\frac{\alpha\Omega}{k}} = 0 \\ k_m &= \sqrt[3]{\frac{\alpha\Omega}{16\eta_T}} \end{aligned}$$

$$\omega_{\rm cyc} = \left(\frac{\alpha\Omega}{2}\right)^{2/3} \left(\frac{1}{4\eta_T}\right)^{1/6}$$

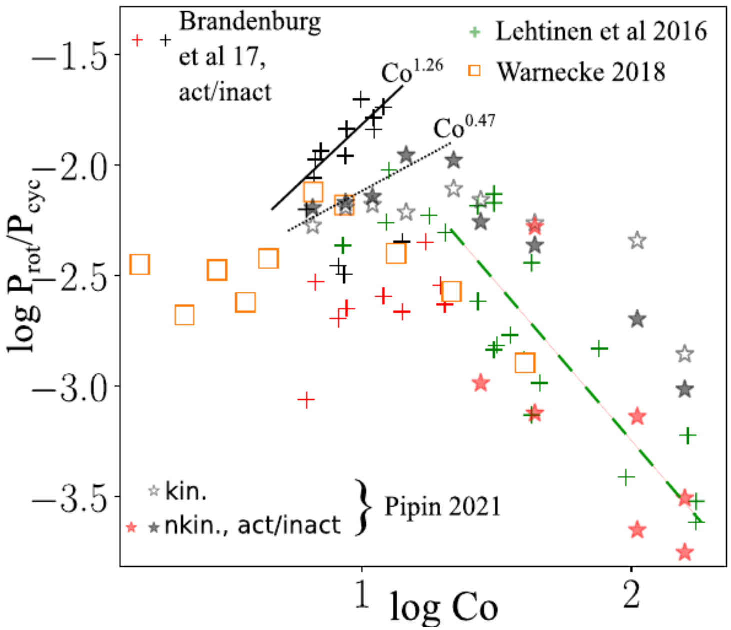
b) Saturation. The efficient dynamo number is close to the critical, $\omega_{\rm cyc} \sim \eta_T k^2$ - the cycle period is determine by the turbulent diffusion and it is independent of Ω.

Dependence of cycle period on stellar rotation rate

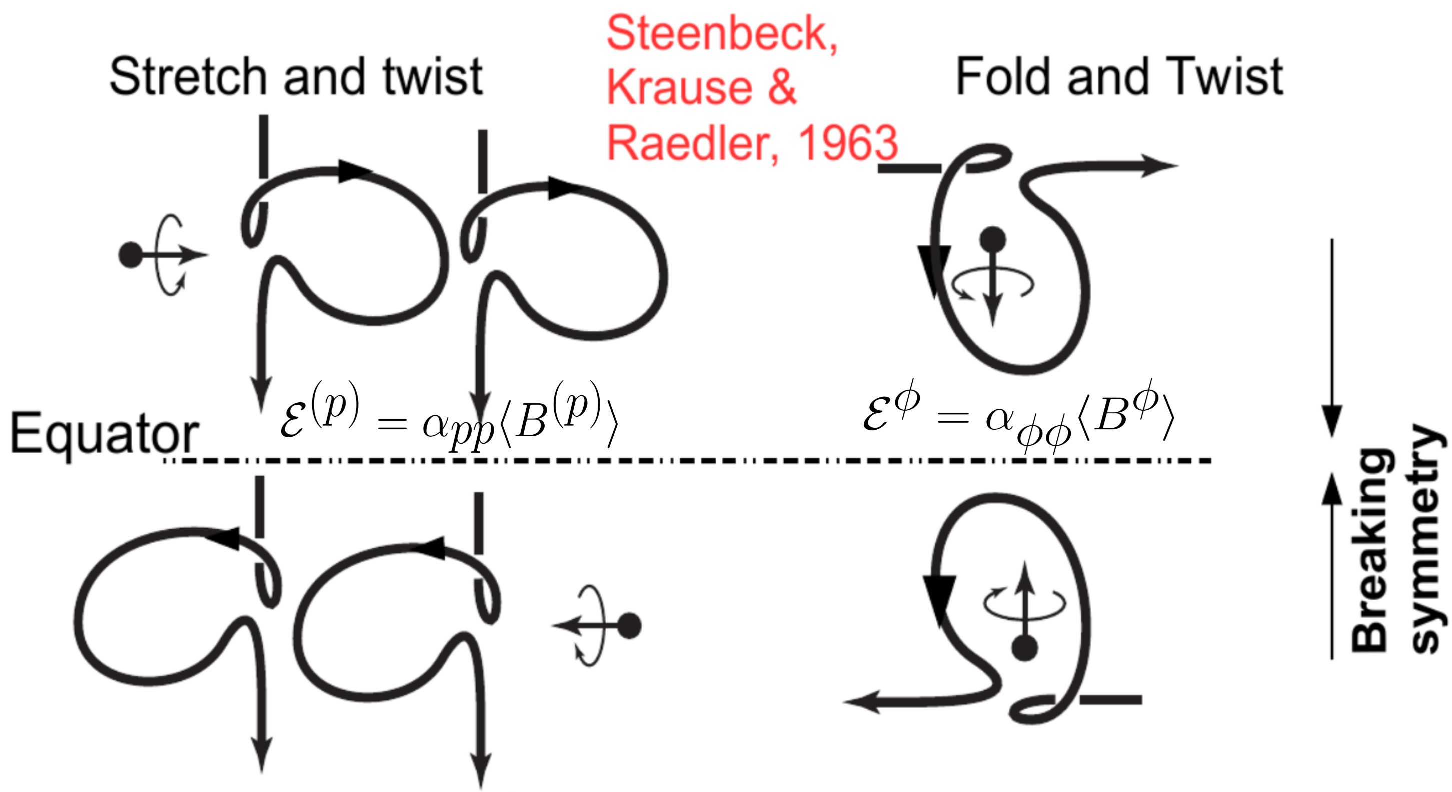
Red and black crosses show the results of Brandenburg et al. (2017), green crosses those of Lehtinen et al. (2016), orange squares the models of Warnecke (2018), and the asterisks are from the models of Pipin (2021);act/inact marks the active and inactive branches of activity, respectively; while kin/nkin stand for kinematic and non-kinematic models, respectively. Note that $\mathrm{Co} = 2\Omega\tau_c$, we employ $\alpha \sim \Omega\tau_c$. So, for the rate:

$$\frac{\omega_{\mathrm{cyc}}}{\omega_{\mathrm{rot}}} \sim \mathrm{Co}^{1/3}{\eta_T}^{-1/6} \quad \text{active}$$

$$\frac{\omega_{\mathrm{cyc}}}{\omega_{\mathrm{rot}}} \sim \mathrm{Co}^{-1} \quad \text{saturated}$$



The α^2 dynamo



Theoretical expectations about α^2 dynamo:

1) Steady. Evidence for the cyclic dynamo from DNS (Brandenburg et al 2015)

2) Toroidal and poloidal components have comparable magnitude (Raedler 1986)

3) Non-axisymmetric magnetic field grows faster than axisymmetric one

The α^2 dynamo

Here, we assume that α effect is isotropic, i.e. $\alpha_{\rm pp} = \alpha_{\varphi\varphi}$. After substitution:

$$\begin{aligned} \frac{\partial \boldsymbol{e}^{\varphi} B}{\partial t} &= \quad \boldsymbol{\nabla} \times \alpha \boldsymbol{\nabla} \times A\, \boldsymbol{e}^{\varphi} - \eta_T \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times B\, \boldsymbol{e}^{\varphi} \\ \frac{\partial \boldsymbol{e}^{\varphi} A}{\partial t} &= \alpha \boldsymbol{e}^{\varphi} B - \eta_T \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times A\, \boldsymbol{e}^{\varphi} \end{aligned}$$

Use the eigen-wave representation: $B, A \sim \exp(\sigma t + \mathrm{i} \boldsymbol{k} \cdot \boldsymbol{r})$. Proceeding in the similar way as previously we get

$$\begin{aligned} \sigma B &= \alpha k^2 A - \eta_T \boldsymbol{k}^2 B \\ \sigma A &= \alpha B - \eta_T \boldsymbol{k}^2 A \end{aligned}$$

Then compatibility condition determine the eigen values:

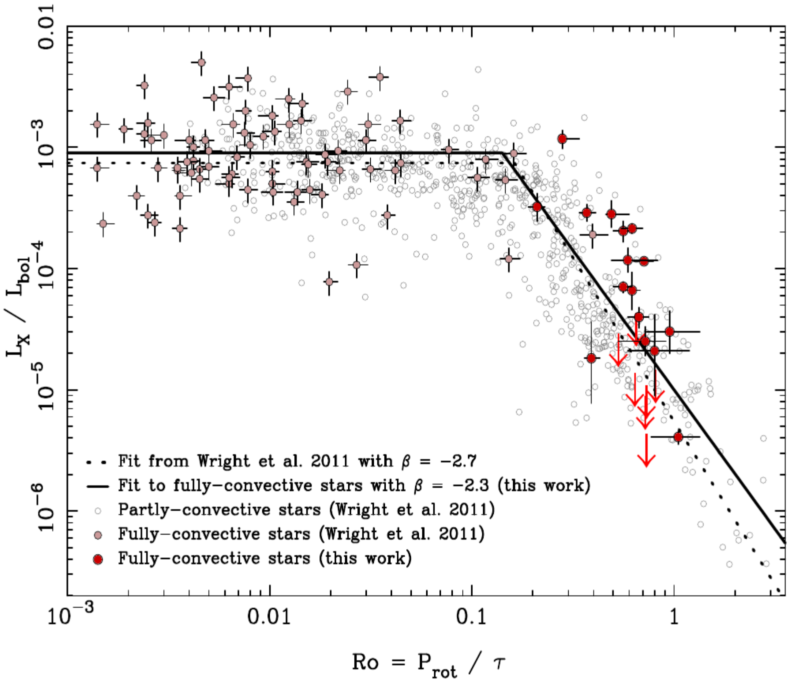
$$\begin{vmatrix} \sigma + \eta_T k^2 & -\alpha k^2 \\ -\alpha & \sigma + \eta_T k^2 \end{vmatrix} = 0$$

The growing modes have $\mathrm{Re}(\sigma) > 0$. The solution of the determinant is

$$\sigma = \alpha k - \eta_T k^2$$

For the growing mode $\alpha > \eta_T k$, no wave, a stationary dynamo. The maximum generation rate wavelength is $k = \alpha / 2\eta_T$.

Stellar activity magnitude, X-ray



For the partially convective stars we can expect $\alpha\Omega$ dynamo and for the fully convective- α^2

In the linear regime the growth rate for the $\alpha\Omega$ dynamo is

$$\gamma_{\alpha\Omega} \sim \left(\frac{\alpha\Omega}{2}\right)^{2/3}\left(\frac{1}{4\eta_T}\right)^{1/6} \sim \Omega^{4/3}$$

and for α^2

$$\gamma_{\alpha 2} \sim \frac{\alpha^2}{2\eta_T} \sim \Omega^2$$

Here we do not take into account $\eta_T(\Omega)$

The large Reynolds number limit. Consider the forced turbulence

$$\frac{\boldsymbol{u}}{\boldsymbol{\tau_c}} \approx \boldsymbol{2u \times \Omega + \frac{u^{(0)}}{\tau_c}\cdots}$$

Let us divide the flow into sum along and perpendicular to the rotation axis:

$$\boldsymbol{u = u_{\perp} + u_{\|}, u_{\perp} = u - \frac{\Omega}{\Omega^2}(u \cdot \Omega); u_{\|} = \frac{\Omega}{\Omega^2}(u \cdot \Omega)}$$

Assume that in the background turbulence (without global rotation) is isotropic, $\langle u_{\perp}^{(0)2} \rangle = \langle u_{\|}^{(0)2} \rangle$. Then

$$(\delta_{\mathrm{ij}} - 2\tau_c \varepsilon_{\mathrm{ijn}} \Omega_n) u_{\perp j} \;=\; u_{\perp i}^{(0)}; u_{\| i} = u_{\| i}^{(0)}$$

Consider intensity of turbulent flows, $\langle u_{\perp}^2 \rangle$ and $\langle u_{\|}^2 \rangle$ under effect of the global rotation:

$$(1 + 4\tau_c^2 \Omega^2) \langle u_{\perp}^2 \rangle \;=\; \langle u_{\perp}^{(0)2} \rangle; \langle u_{\|}^2 \rangle = \langle u_{\|}^{(0)2} \rangle$$

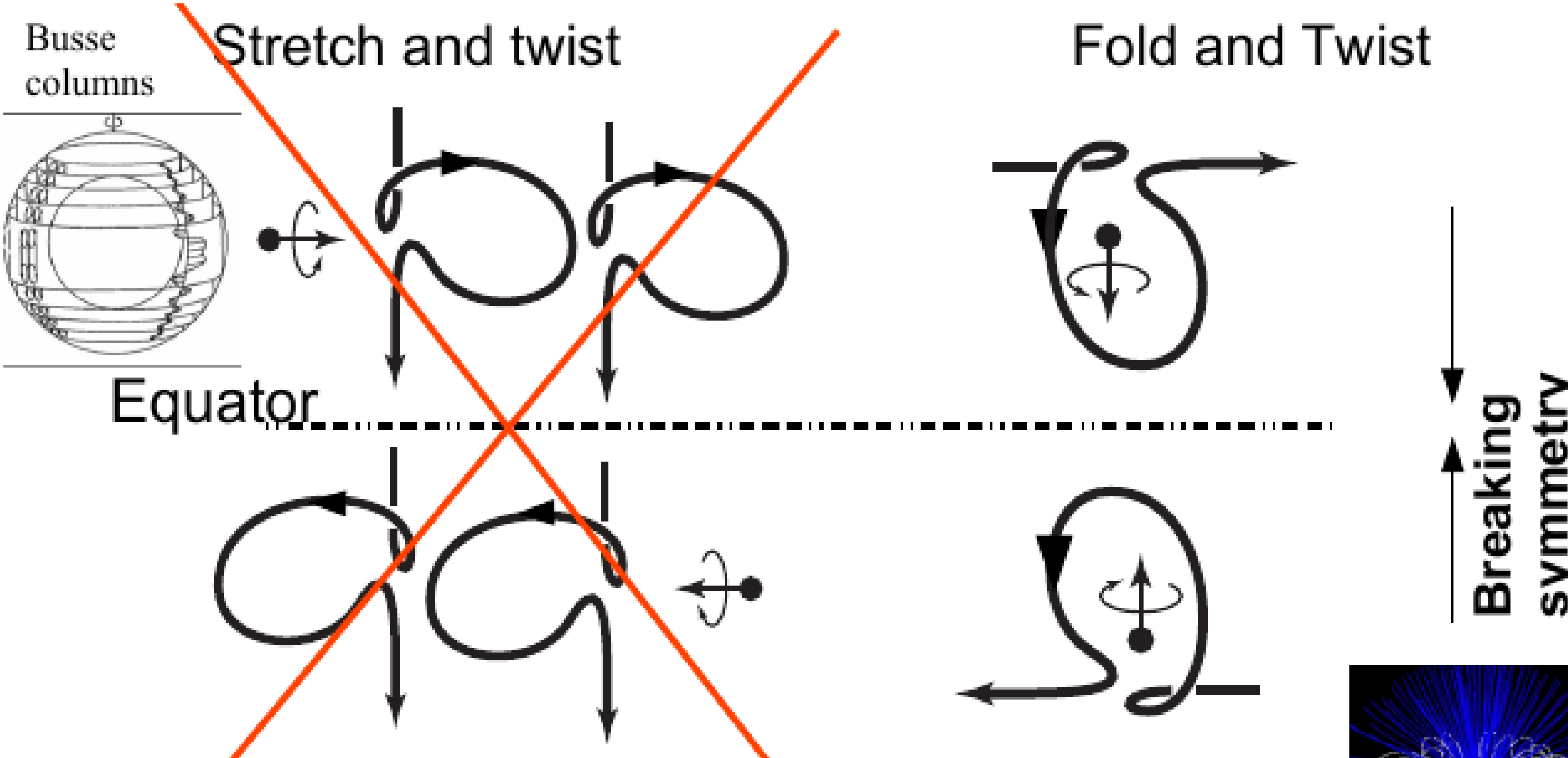
Here we again employ identity $\varepsilon_{\mathrm{ijn}} u_{\perp j} \Omega_n \varepsilon_{\mathrm{ipm}} u_{\perp p} \Omega_m = (\delta_{\mathrm{jp}} \delta_{\mathrm{nm}} - \delta_{\mathrm{jm}} \delta_{\mathrm{np}}) u_{\perp j} \Omega_n u_{\perp p} \Omega_m$

Effect of rotation on the turbulence

With increase of the rotation rate the turbulence become highly anisotropic

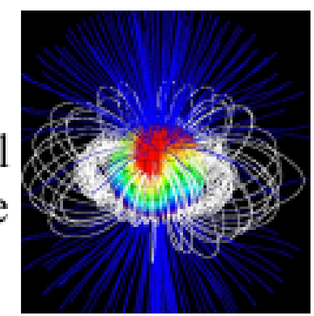
- The mixing in direction which is perpendicular to rotation axis become less turbulent. The turbulent diffusivity and viscosity become anisotropic.
- The heat flux is anisotropic, the polar regions are heated up
- The α effect is anisotropic and it is suppressed along the rotation axis (Kitchatinov & Ruediger 1993)

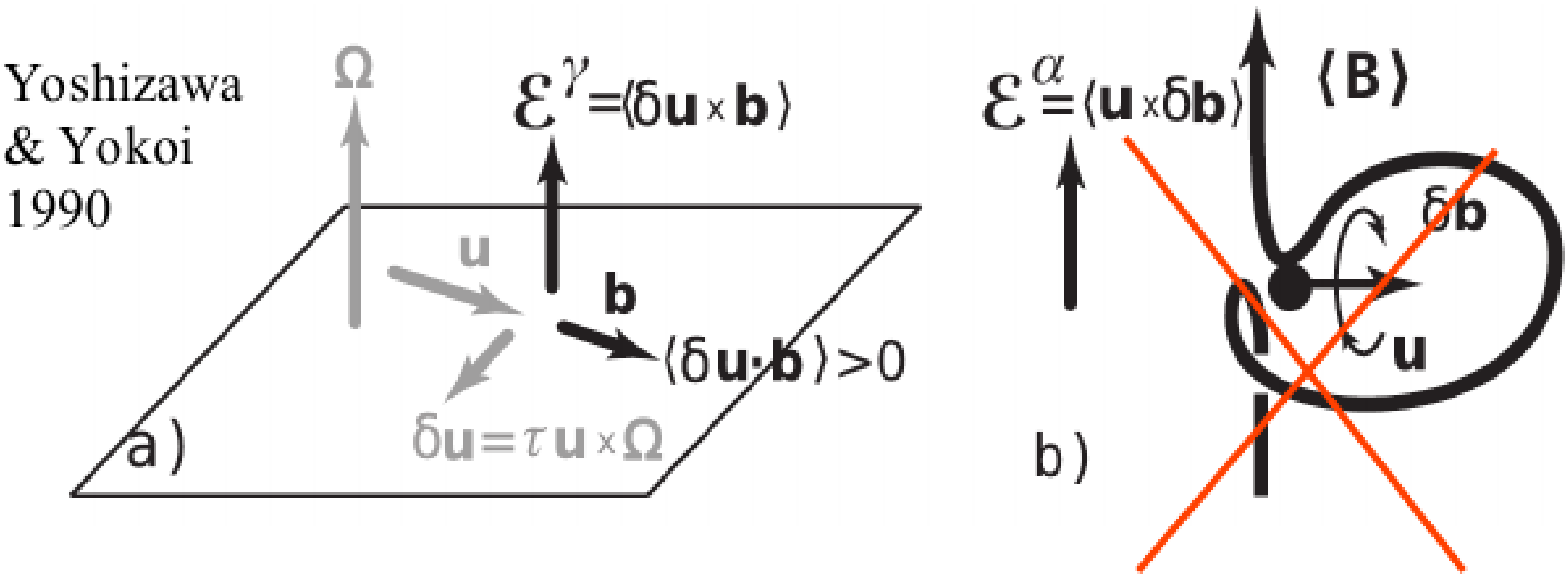
α^2 dynamo is not efficient for the fast rotators



The Coriolis force suppress turbulent motion across the axis of rotation (Busse columns), disabling α^2 generation of the axisymmetric toroidal field

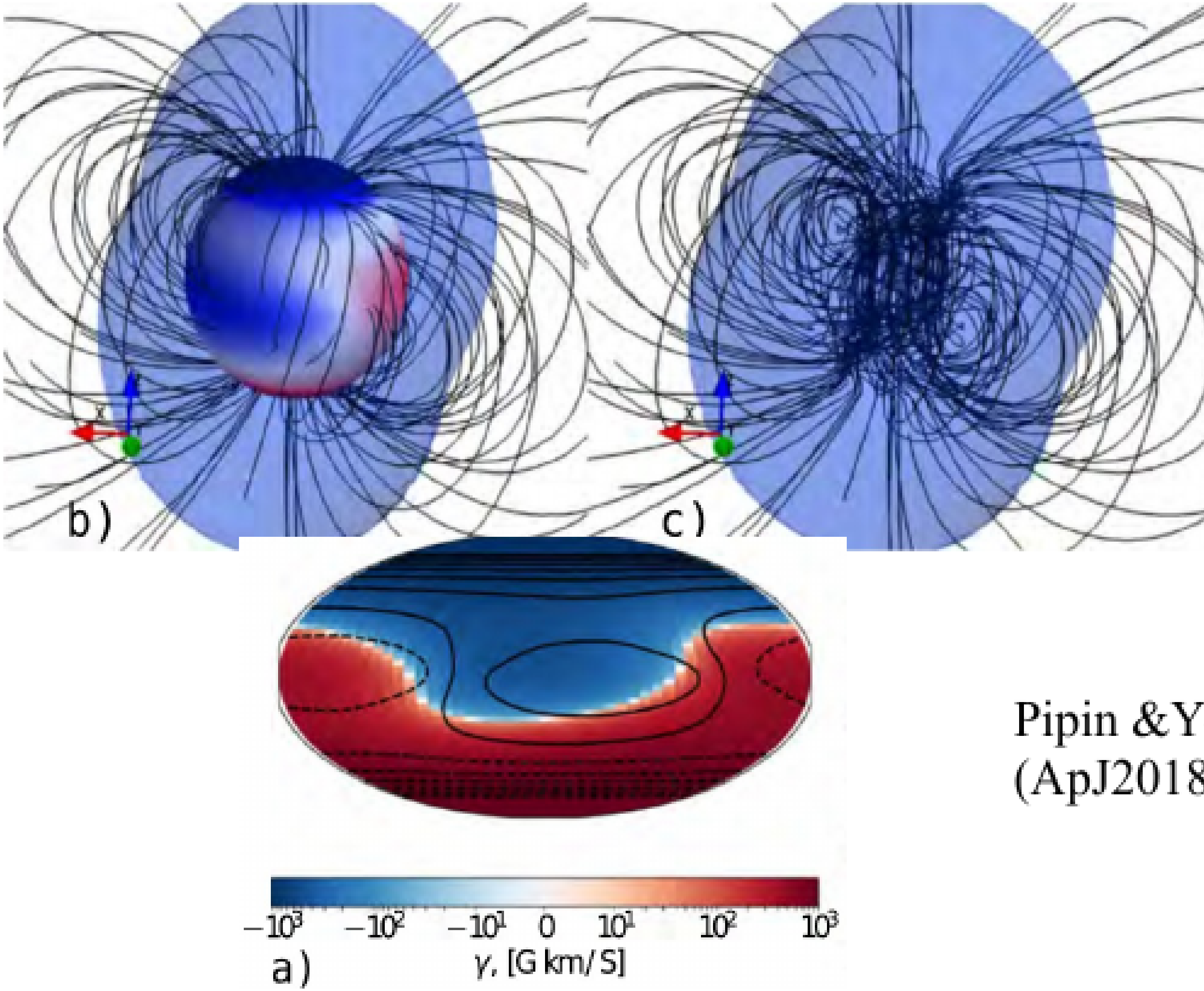
How to generate toroidal magnetic field in absence of differetial rotation?





Cross-helicity effect may works near upper boundary of convection zone. For the Sun may be less important because spots are small.

Example of cross-helicity dynamo model



Pipin &Yokoi
(ApJ2018)

Summary

- The dynamo is a an instability of the large-scale magnetic field.
- The astrophysical objects demonstrate numerous dynamo scenarios
  - Cyclic dynamos are typical for the partially convective stars. Here the dynamo is due to the large-scale flow (differential rotation generate B^{φ}) and turbulent generation, e.g., α effect, generate B^{p}, $|B^{\varphi}| \gg |B^{p}|$.
  - Steady dynamos are typical for the fully convective stars and planet as well. Here, $|B^{\varphi}| \sim |B^{p}|$ these objects are examples of turbulent dynamos: α^2, or $\alpha^2\Gamma$
- The global rotation results to the anisotropy of the turbulent effects
  - anisotropy of α effect, the axisymmetric α^2 dynamo is suppressed,
  - anisotropy of turbulent diffusion and the heat flux