# Mean-field solar dynamo models

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# Topics:

- Derivation
- Dynamo models in the spherical shell
- Eigen value problem
- Basic properties
- Nonlinear effects

# Derivation

We consider the one-fluid ideal magnetohydrodynamics equations

$$
\begin{aligned}
\frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}), \\
\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} &= \boldsymbol{\nabla} p + \rho \boldsymbol{g} + \frac{1}{4\pi}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}, \\
\rho T\left(\frac{\partial s}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})s\right) &= \boldsymbol{\nabla} \cdot (\boldsymbol{F}^c + \boldsymbol{F}^{\mathrm{rad}}) + \varepsilon^s
\end{aligned}
$$

Then decompose for the sum of the mean and fluctuating parts:

$$
\boldsymbol{v} = \langle \boldsymbol{U} \rangle + \boldsymbol{u};\, \boldsymbol{B} = \langle \boldsymbol{B} \rangle + \boldsymbol{b};\, s = \langle s \rangle + s', \ldots
$$

Assume $\partial_{\boldsymbol{x}} \boldsymbol{b} \sim \dfrac{\boldsymbol{b}}{\ell_c}$ and $\partial_{\boldsymbol{x}} \langle \boldsymbol{B} \rangle \sim \dfrac{\langle \boldsymbol{B} \rangle}{L_a}$, and the scale separation $\ell_c \ll L_a$.Then the set of vectors to construct $\langle \mathbf{u} \times \mathbf{b} \rangle$ may consists of

i. the large-scale field: $\langle \boldsymbol{B} \rangle$, $\nabla \times \langle \boldsymbol{B} \rangle$ and perhaps $\boldsymbol{\Omega}, \nabla \times \langle \boldsymbol{U} \rangle$, $\boldsymbol{\nabla} \bar{\rho}$

ii. the average effects of the turbulent fields: $\boldsymbol{\nabla} \langle u^{(0)2} \rangle$, $\boldsymbol{\nabla} \langle b^{(0)2} \rangle$, etc.

# Equations driving the turbulence:

The governing equations for fluctuating magnetic field and velocity are written in a rotating coordinate system as follows:

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times \left(\mathbf{u} \times \overline{\mathbf{B}} + \overline{\mathbf{V}} \times \mathbf{b}\right) + \eta \nabla^2 \mathbf{b} + \mathfrak{G}, \tag{1}$$

$$\frac{\partial m_i}{\partial t} + 2(\boldsymbol{\Omega} \times \mathbf{m})_i = -\nabla_i \left( p - \frac{2}{3}(\mathbf{G} \cdot \mathbf{m})\nu + \frac{(\mathbf{b} \cdot \overline{\mathbf{B}})}{2\mu} \right) + \nu \Delta m_i + \nu (\mathbf{G} \cdot \nabla) m_i$$
$$+ \frac{1}{\mu} \nabla_j \left( \overline{B}_j b_i + \overline{B}_i b_j \right) - \nabla_j \left( \overline{V}_j m_i + \overline{V}_i m_j \right) + f_i + \mathfrak{F}_i, \tag{2}$$

where $\mathfrak{G}$, $\mathfrak{F}$ are nonlinear contributions of fluctuating fields, $\mathbf{m} = \bar{\rho}\mathbf{u}$, $\mathbf{G} = \nabla \log \bar{\rho}$ is the density stratification scale of the media, $p$ is the fluctuating pressure, $\boldsymbol{\Omega}$ is the angular velocity responsible for the Coriolis force, $\overline{\mathbf{V}}$ is mean flow which is a weakly variable in space, $\mathbf{f}$ is the random force driving the turbulence.

Statistics of the random forces, and, governing the turbulence is given. We solve these equations forming the equations for the second order correlations and using the scale separation approximation (see analytic results Pipin2008).

To compute $\mathcal{E}$ it is convenient to write the MHD equations in Fourier space:

$$
\begin{aligned}
\left(\frac{\partial}{\partial t}+\eta z^2\right)\hat{b}_j &= \mathrm{i}z_l\int\left[\widehat{m}_j(\mathbf{z}-\mathbf{q})\left(\frac{\hat{\overline{\overline{B}}}_l}{\rho}\right)(\mathbf{q})-\widehat{m}_l(\mathbf{z}-\mathbf{q})\left(\frac{\hat{\overline{\overline{B}}}_j}{\rho}\right)(\mathbf{q})\right]\mathrm{d}\mathbf{q} \\
&+ \mathrm{i}z_l\int\left[\widehat{b}_l(\mathbf{z}-\mathbf{q})\hat{\overline{\overline{V}}}_j(\mathbf{q})-\widehat{b}_j(\mathbf{z}-\mathbf{q})\hat{\overline{\overline{V}}}_l(\mathbf{q})\right]\mathrm{d}\mathbf{q}+\widehat{\mathfrak{G}}_j. \\
\left(\frac{\partial}{\partial t}+\nu z^2+\mathrm{i}\nu\left(\mathbf{G}\mathbf{z}\right)\right)\hat{m}_i &= \hat{f}_i+\hat{\mathfrak{F}}_i-2\left(\mathbf{\Omega}\hat{\mathbf{z}}\right)\left(\hat{\mathbf{z}}\times\hat{\mathbf{m}}\right)_i \qquad (2) \\
&- \mathrm{i}\pi_{if}(\mathbf{z})z_l\int\left[\widehat{m}_l(\mathbf{z}-\mathbf{q})\hat{\overline{\overline{V}}}_f(\mathbf{q})+\widehat{m}_f(\mathbf{z}-\mathbf{q})\hat{\overline{\overline{V}}}_l(\mathbf{q})\right]\mathrm{d}\mathbf{q} \\
&+ \frac{\mathrm{i}}{\mu}\pi_{if}(\mathbf{z})z_l\int\left[\widehat{b}_l(\mathbf{z}-\mathbf{q})\hat{\overline{\overline{B}}}_f\left(\mathbf{q}\right)+\widehat{b}_f(\mathbf{z}-\mathbf{q})\hat{\overline{\overline{B}}}_l\left(q\right)\right]\mathrm{d}\mathbf{q},
\end{aligned}
$$

# Equations to solve: the second order correlations in Fourier space

$$\frac{\partial}{\partial t}\left\langle \hat{m}_i(\mathbf{z})\hat{b}_j(\mathbf{z}')\right\rangle = Th^{\varkappa}_{ij}(\mathbf{z},\mathbf{z}') - \left(\eta z'^2 + \nu z^2 + \mathrm{i}\nu(\mathbf{G}\cdot\mathbf{z})\right)\left\langle \hat{m}_i(\mathbf{z})\hat{b}_j(\mathbf{z}')\right\rangle$$
$$\times\, \mathrm{i}z'_l \int\left[\langle \hat{m}_i(\mathbf{z})\hat{m}_j(\mathbf{z}'-\mathbf{q})\rangle\left(\frac{\hat{\overline{B}}_l}{\rho}\right)(\mathbf{q}) - \langle \hat{m}_i(\mathbf{z})\hat{m}_l(\mathbf{z}'-\mathbf{q})\rangle\left(\frac{\hat{\overline{B}}_j}{\rho}\right)(\mathbf{q})\right]\mathrm{d}\mathbf{q}$$
$$- 2(\mathbf{\Omega}\cdot\hat{\mathbf{z}})\varepsilon_{iln}\hat{z}_l\left\langle \hat{m}_n(\mathbf{z})\hat{b}_j(\mathbf{z}')\right\rangle + \mathrm{i}z'_l\int\left[\left\langle \hat{m}_i(\mathbf{z})\widehat{b_l}(\mathbf{z}'-\mathbf{q})\right\rangle\hat{\overline{V}}_j(\mathbf{q})\right.$$
$$\left. - \left\langle \hat{m}_i(\mathbf{z})\widehat{b_j}(\mathbf{z}'-\mathbf{q})\right\rangle\hat{\overline{V}}_l(\mathbf{q})\right]\mathrm{d}\mathbf{q}$$
$$- \mathrm{i}\pi_{if}(\mathbf{z})z_l\int\left[\left\langle \widehat{m_l}(\mathbf{z}-\mathbf{q})\hat{b}_j(\mathbf{z}')\right\rangle\hat{\overline{V}}_f(\mathbf{q}) + \left\langle \widehat{m_f}(\mathbf{z}-\mathbf{q})\hat{b}_j(\mathbf{z}')\right\rangle\hat{\overline{V}}_l(\mathbf{q})\right]\mathrm{d}\mathbf{q}$$
$$+ \frac{\mathrm{i}}{\mu}z_l\pi_{if}(\mathbf{z})\int\left[\left\langle \widehat{b_l}(\mathbf{z}-\mathbf{q})\hat{b}_j(\mathbf{z}')\right\rangle\overline{B}_f(\mathbf{q}) + \left\langle \widehat{b_f}(\mathbf{z}-\mathbf{q})\hat{b}_j(\mathbf{z}')\right\rangle\overline{B}_l(q)\right]\mathrm{d}\mathbf{q}, \qquad (5)$$

Here, is the evolution equation for corelation of the fluctuating momentum $\hat{\boldsymbol{m}} = \int \bar{\rho}\boldsymbol{u}e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$, and fluctuating magnetic field $\hat{\boldsymbol{b}} = \int \boldsymbol{b}e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\mathrm{d}\boldsymbol{x}$ , $\mathrm{Th_{ij}}$ is the third order correlation. Note, that, the turbulent parameters are in convolution with the large-scale field $\boldsymbol{B}$ and $\boldsymbol{V}$

# Tau approximation

Minimal tau-approximation, it is valid for Rm, Re $\gg$ 1

$$\mathrm{Th}_{\mathrm{ij}}^{(\varkappa)} \;\rightarrow\; -\frac{\langle \hat{m}_i(\boldsymbol{z})\hat{b}_j(\boldsymbol{z}')\rangle}{\tau_c},$$

$$\mathrm{Th}_{\mathrm{ij}}^{(\nu)} \;\rightarrow\; -\frac{\langle \hat{m}_i(\boldsymbol{z})\hat{m}_j(\boldsymbol{z}')\rangle - \langle \hat{m}_i(\boldsymbol{z})\hat{m}_j(\boldsymbol{z}')\rangle^{(0)}}{\tau_c},$$

$$\mathrm{Th}_{\mathrm{ij}}^{(h)} \;\rightarrow\; -\frac{\langle \hat{b}_i(\boldsymbol{z})\hat{b}_j(\boldsymbol{z}')\rangle - \langle \hat{b}_i(\boldsymbol{z})\hat{b}_j(\boldsymbol{z}')\rangle^{(0)}}{\tau_c},$$

It is assumed that the third-order moments relax to the second order ones

# Dynamo equations

$$\frac{\partial\langle \boldsymbol{B}\rangle}{\partial t} = \boldsymbol{\nabla}\times\left(\boldsymbol{\mathcal{E}} + \langle \boldsymbol{U}\rangle \times \langle \boldsymbol{B}\rangle\right), \boldsymbol{\mathcal{E}} = \langle \boldsymbol{u}\times\boldsymbol{b}\rangle$$

The expression for $\mathcal{E}$ is compatible with scale separation approximation and symmetry of MHD fields (Krause&Raedler 1980):

$$\mathcal{E}_i = (\alpha_{\mathrm{ij}} + \gamma_{\mathrm{ij}})\langle B_j\rangle + \eta_{\mathrm{ijk}}\nabla_j\langle B_k\rangle + \dots + (\Omega\times J, W\times J, \kappa, \Gamma)$$

Axisymmetric magnetic fields

$$\langle \boldsymbol{B}\rangle = \boldsymbol{B}^{(t)} + \boldsymbol{B}^{(p)} = \boldsymbol{e}_{\varphi}B + \boldsymbol{\nabla}\times\frac{A}{r\sin\theta}\boldsymbol{e}_{\varphi}; \langle \boldsymbol{U}\rangle = \boldsymbol{e}_{\varphi}r\sin\theta + \langle \boldsymbol{U}^{(p)}\rangle; \langle \boldsymbol{U}^{(p)}\rangle = \frac{1}{\bar{\rho}}\boldsymbol{\nabla}\times\frac{\psi}{r\sin\theta}\boldsymbol{e}_{\varphi}$$

After substitution into induction equations we get

$$\begin{aligned}
\frac{\partial B}{\partial t} &= \frac{1}{r}\frac{\partial(\Omega, A)}{\partial(r,\theta)} + \frac{1}{r}\left(\frac{\partial r(\mathcal{E}_\theta - \langle U_r\rangle B)}{\partial r} - \frac{\partial(\mathcal{E}_r + \langle U_\theta\rangle B)}{\partial\theta}\right),\\
\frac{\partial A}{\partial t} &= r\sin\theta\mathcal{E}_\varphi - \frac{\langle U_\theta\rangle}{r}\frac{\partial A}{\partial\theta} - \langle U_r\rangle\frac{\partial A}{\partial r},
\end{aligned}$$

# Mean electromotive force

$$\mathcal{E}_i = (\alpha_{\mathrm{ij}} + \gamma_{\mathrm{ij}})\langle B_j\rangle + \eta_{\mathrm{ijk}}\nabla_j\langle B_k\rangle$$

$$\alpha_{\mathrm{ij}} = \alpha^H_{\mathrm{ij}}\psi_\alpha(\beta) + \alpha^M_{\mathrm{ij}}\psi_\alpha(\beta)\frac{\langle \boldsymbol{a}\cdot\boldsymbol{b}\rangle\tau_c}{4\pi\bar{\rho}\ell_c^2} ,$$

$$\gamma_{ij} = \gamma_{ij}^{(\Lambda\rho)} + \frac{\alpha_{\mathrm{MLT}} u_{\mathrm{rms}}}{\gamma}\mathcal{H}(\beta)\,\hat{\mathrm{r}}_{\mathrm{n}}\varepsilon_{\mathrm{inj}},$$

$$\gamma_{ij}^{(\Lambda\rho)} = 3\nu_T f_1^{(a)}\left\{\left(\boldsymbol{\Omega}\cdot\boldsymbol{\Lambda}^{(\rho)}\right)\frac{\Omega_n}{\Omega^2}\varepsilon_{\mathrm{inj}} - \frac{\Omega_j}{\Omega^2}\varepsilon_{inm}\Omega_n\Lambda_m^{(\rho)}\right\},$$

$$\alpha_{ij}^{(H)} = \delta_{ij}\left\{f_{10}^{(a)}\left[\boldsymbol{e}\cdot\boldsymbol{\Lambda}^{(\rho)}\right] + f_{11}^{(a)}\left[\boldsymbol{e}\cdot\boldsymbol{\Lambda}^{(u)}\right]\right\} + e_ie_j\left\{f_5^{(a)}\left[\boldsymbol{e}\cdot\boldsymbol{\Lambda}^{(\rho)}\right] + f_4^{(a)}\left[\boldsymbol{e}\cdot\boldsymbol{\Lambda}^{(u)}\right]\right\} + \left\{\left[e_i\Lambda_j^{(\rho)} + e_j\Lambda_i^{(\rho)}\right]f_6^{(a)} + \left[e_i\Lambda_j^{(u)} + e_j\Lambda_i^{(u)}\right]f_8^{(a)}\right\}, \quad \text{(A2)}$$

where $\boldsymbol{e} = \boldsymbol{\Omega}/\Omega$, $\boldsymbol{\Lambda}^{(\rho)} = \boldsymbol{\nabla}\log\overline{\rho}$, $\boldsymbol{\Lambda}^{(u)} = \boldsymbol{\nabla}\log(u'\ell)$ and $\alpha_{ij}^{(M)}$ is

$$\alpha_{ij}^{(M)} = 2f_2^{(a)}\delta_{ij} - 2f_1^{(a)}e_ie_j. \quad \text{(A3)}$$

Functions $f_n^{(a)}(\Omega^*)$ were defined by Pipin (2008), $\Omega^* = 2\tau_c\Omega_0$,

$$\eta_{\mathrm{ijk}} = 3\eta_T\left\{\left(2f_1^{(a)} - f_2^{(d)}\right)\varepsilon_{\mathrm{ijk}} + 2f_1^{(a)}\frac{\Omega_i\Omega_n}{\Omega^2}\varepsilon_{\mathrm{jnk}}\right\}$$

$$\boldsymbol{\mathcal{E}}^{(\eta)} = 3\eta_T\left(2f_1^{(a)} - f_2^{(d)}\right)\langle\boldsymbol{J}\rangle + 6\eta_T f_1^{(a)}\boldsymbol{\Omega}\frac{\boldsymbol{\Omega}\cdot\langle\boldsymbol{J}\rangle}{\Omega^2},\ \eta_T = \frac{\langle u^{(0)2}\rangle}{3}\tau_c$$

# Distribution of the dynamo parameters in the SCZ

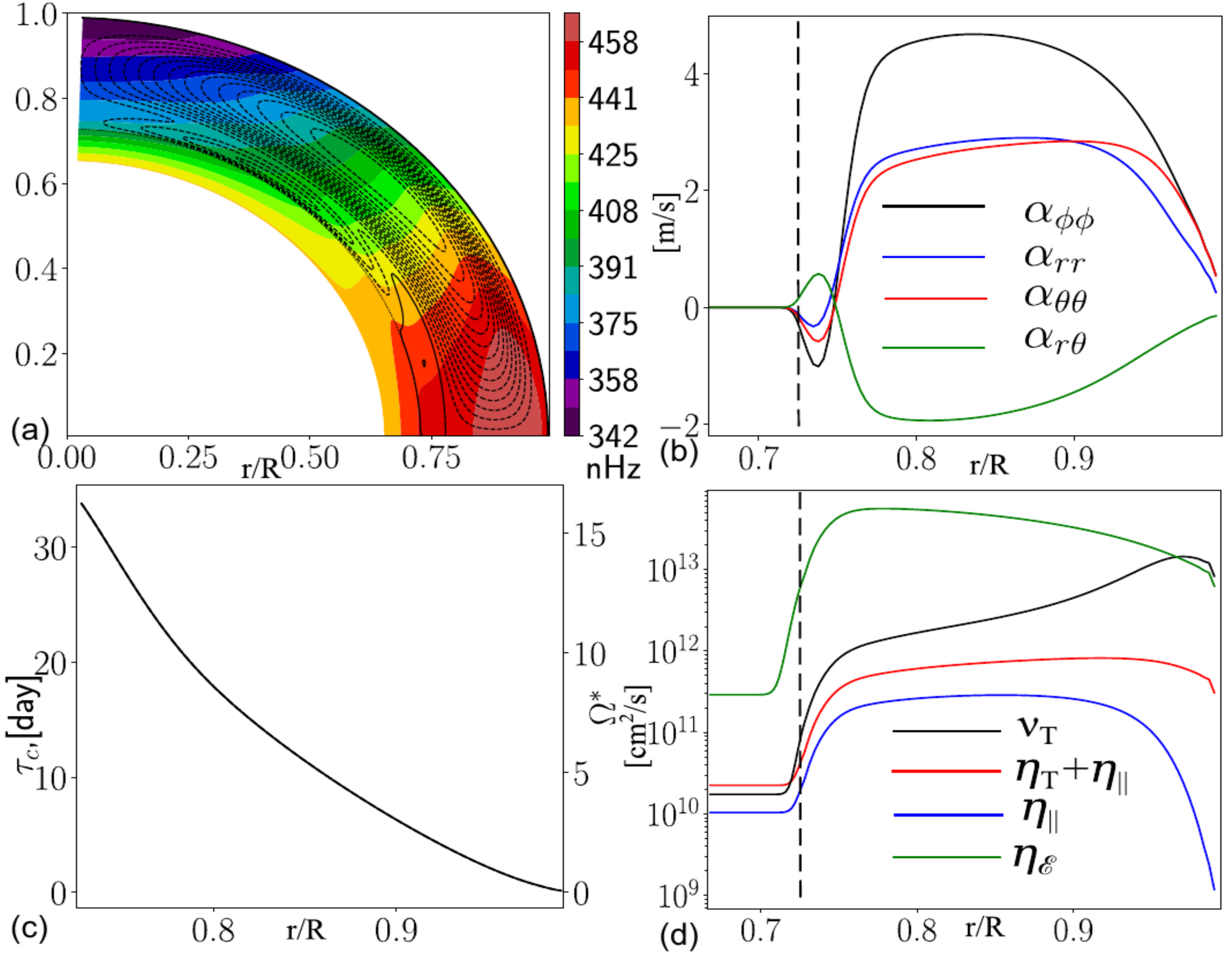
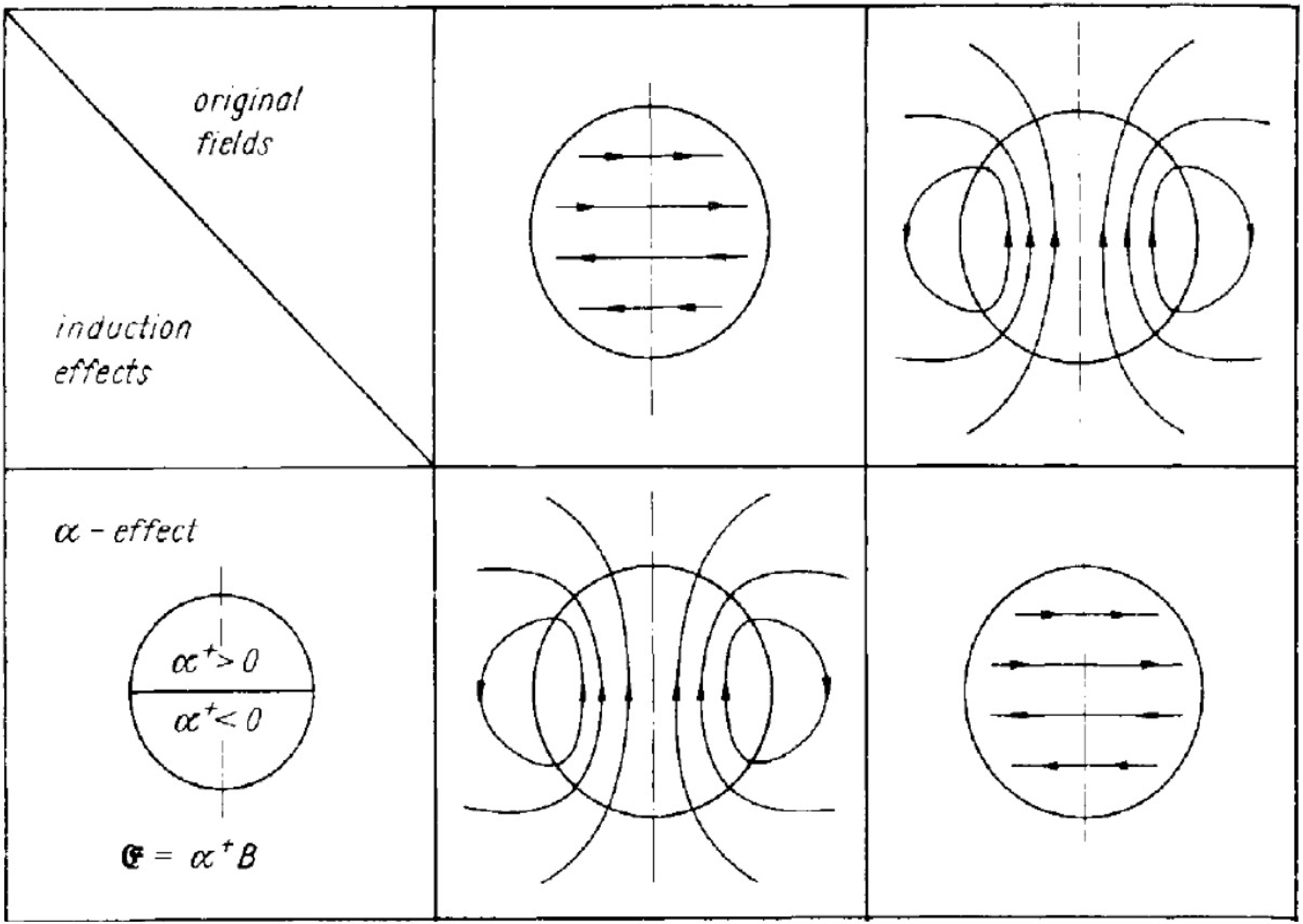


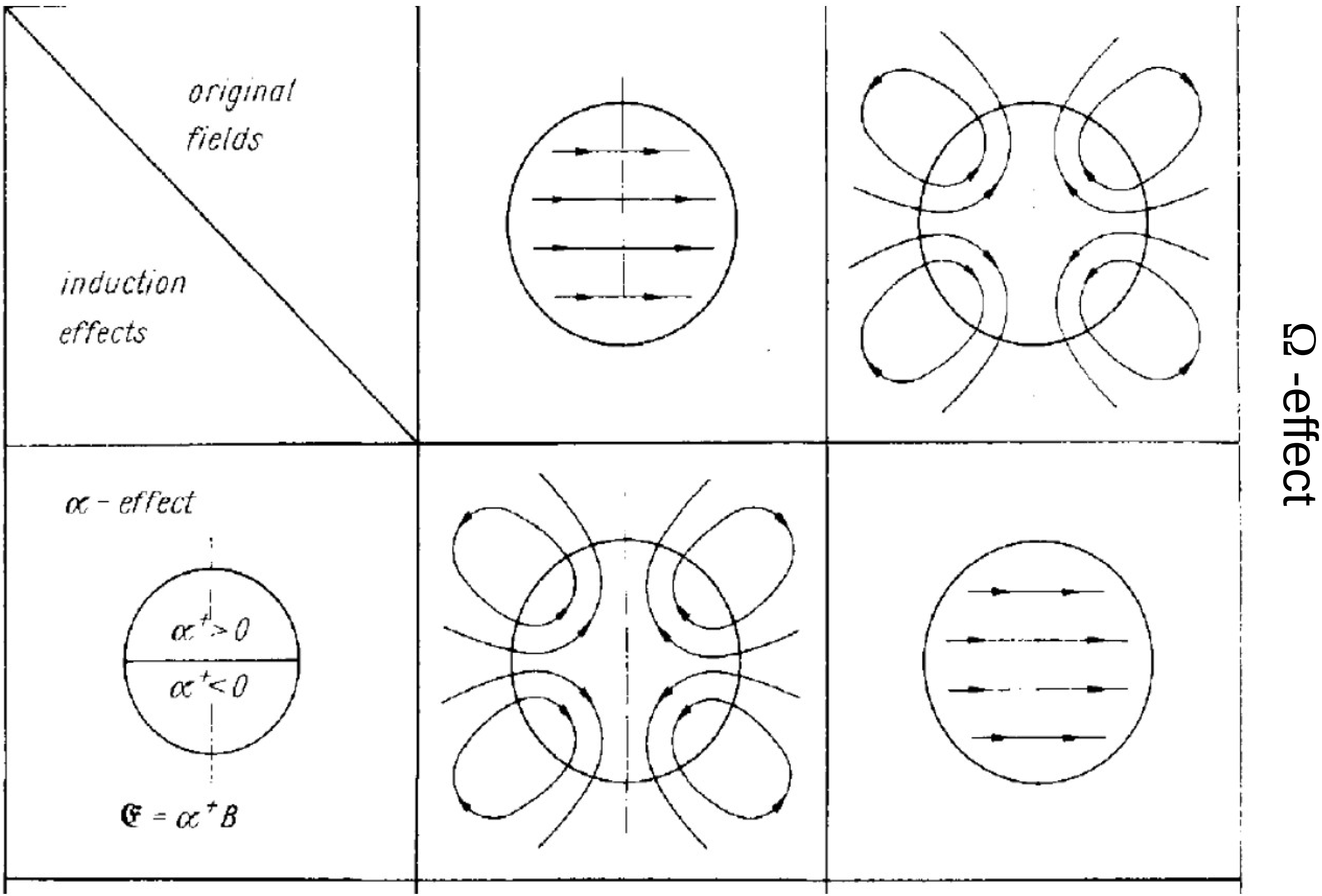

**Figure 1.** (a) The meridional circulation (streamlines) and the angular velocity distributions; the magnitude of circulation velocity is of 13 ms$^{-1}$ on the surface at the latitude of 45°; (b) the $\alpha$-effect tensor distributions at the latitude of 45°, the dash line shows the convection zone boundary; (c) radial profiles of the convective turnover time, $\tau_c$, (left $y$-axis) and the Coriolis number, $\Omega^*$, (right $y$-axis); (d) radial dependencies of the total, $\eta_T + \eta_{||}$, and the rotational induced part, $\eta_{||}$, of the eddy magnetic diffusivity, the eddy viscosity profile, $\nu_T$ and the the $\overline{\mathcal{E}}$ diffusivity profile for $a_E = 1$; hereafter we employ NUMPY/SCIPY (Harris et al. 2020; Virtanen et al. 2020) together with MATPLOTLIB (HUNTER 2007) for post-processing and visualization.

# Dynamo modes, A0




Ω -effect

# Dynamo modes, S0



# 1D models

# Simple model, waves on the spherical surface

To elucidate basic properties of the axisymmetric dynamo, we consider a reduced dynamo model in which the radial dependence of the magnetic field is disregarded, and the same for the flow. In this case, the induction vector of the large-scale magnetic field is represented in terms of the scalar functions as follows:

$$\langle \boldsymbol{B} \rangle = \boldsymbol{B}^{(t)} + \boldsymbol{B}^{(p)} = \boldsymbol{e}_{\varphi} B + \frac{\hat{\boldsymbol{r}}}{R^2 \sin\theta} \frac{\partial \sin\theta A}{\partial \theta} - \frac{\hat{\theta}}{R} A;$$

Consider the isotropic $\alpha$ effect and diffusivity, and write equations for some radial layer $r = R$:

$$\begin{aligned} \frac{\partial B}{\partial t} &= -\sin\theta \frac{\partial \Omega}{\partial r} \frac{\partial (\sin\theta A)}{\partial \mu} + \eta_T \frac{\sin^2\theta}{R^2} \frac{\partial^2 (\sin\theta B)}{\partial^2 \mu} - \frac{B}{\tau_r}, \\ \frac{\partial A}{\partial t} &= \alpha_0 \psi(B) \mu B + \eta_T \frac{\sin^2\theta}{R^2} \frac{\partial^2 (\sin\theta A)}{\partial^2 \mu} - \frac{A}{\tau_r}, \psi(B) = \frac{1}{1 + B^2} \end{aligned}$$

See Noeys et al 1984, Jennings et al 1990, Moss et al 2008

# 1D model (Jennings et al 1990, Kitchatinov et al 1994)

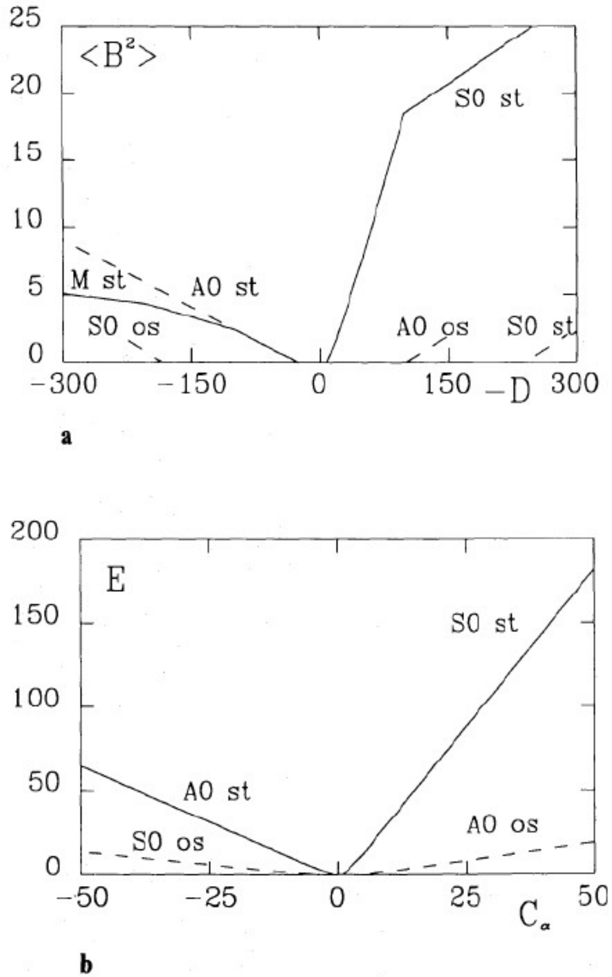
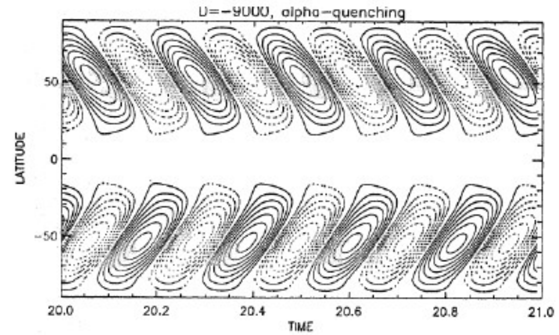
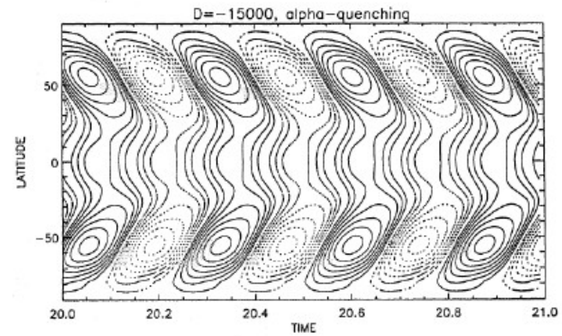



**Fig. 1a and b.** Bifurcation diagram for a 1-D dynamo model without radial extension **a** without curvature and **b** with curvature. Stationary solutions are denoted by "st" and oscillatory ones by "os"

$$\frac{\partial B}{\partial T} = -\sin\theta \mathcal{D}\frac{\partial(\sin\theta A)}{\partial \mu} + \frac{\partial^2(\sin\theta B)}{\partial^2 \mu},$$
$$\frac{\partial A}{\partial T} = C_\alpha \psi(B)\mu B + \sin^2\theta\frac{\partial^2(\sin\theta A)}{\partial^2 \mu},$$

where, we $A/R \rightarrow A$ and define $D = C_\alpha C_\Omega$, $C_\alpha = \frac{\alpha_0 R}{\eta_T}$, and $C_\Omega = \frac{R^3}{\eta_T}\frac{\partial \Omega}{\partial r}$.



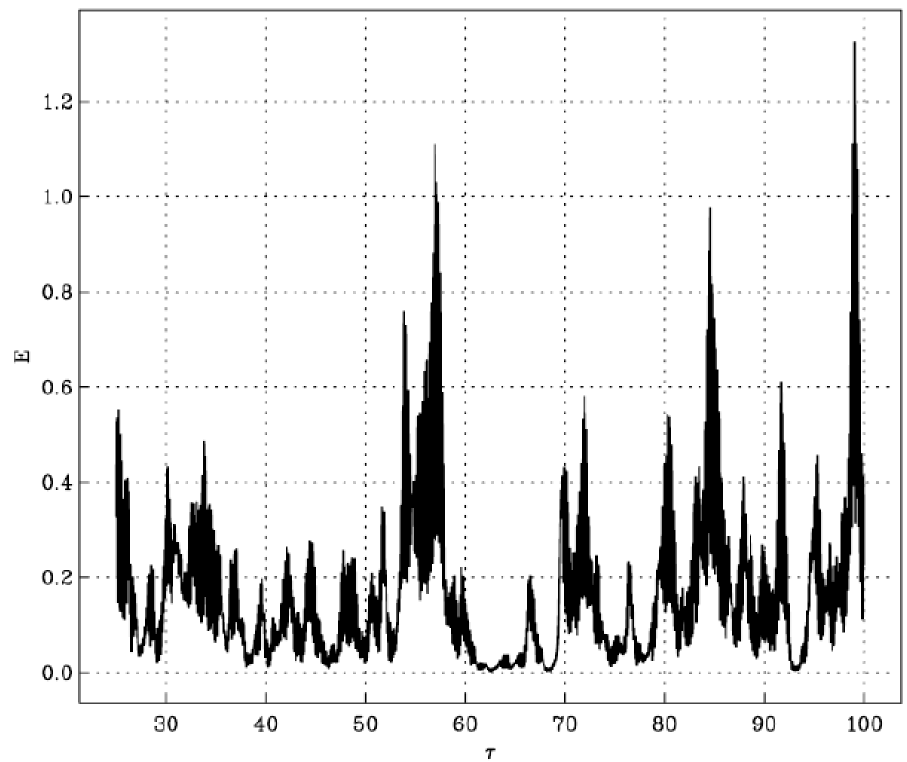
# The long-term cycle dynamics with fluctuating

$$\frac{\partial B}{\partial T} = -\sin\theta\mathcal{D}\frac{\partial(\sin\theta A)}{\partial\mu} + \frac{\partial^2(\sin\theta B)}{\partial^2\mu},$$
$$\frac{\partial A}{\partial T} = \psi(B)(1+\xi)\mu B + \sin^2\theta\frac{\partial^2(\sin\theta A)}{\partial^2\mu}$$

where $\xi$ fluctuates randomly with relative magnitude $\sigma(\xi) = 0.12$.

The Mounder-like minims found if the time-scale of $\xi$ is about one dynamo cycle, i.e, $\sim$10 years.

Moss et all 2008



# Issues of the ”no r” dynamo models

For dynamos in the thin shells radial diffusion will eventually dominate over the latitudinal diffusion because the scale of the mean-field is smaller in radial direction than in the latitudinal one. The 1D models fail to reflect this, the replacement $\eta_T \frac{\partial^2 B}{\partial r^2} \rightarrow -\frac{B}{\tau}$ reduce a key physical process to a single parameter.

# Axisymmetric vs nonaxisymmetric dynamo

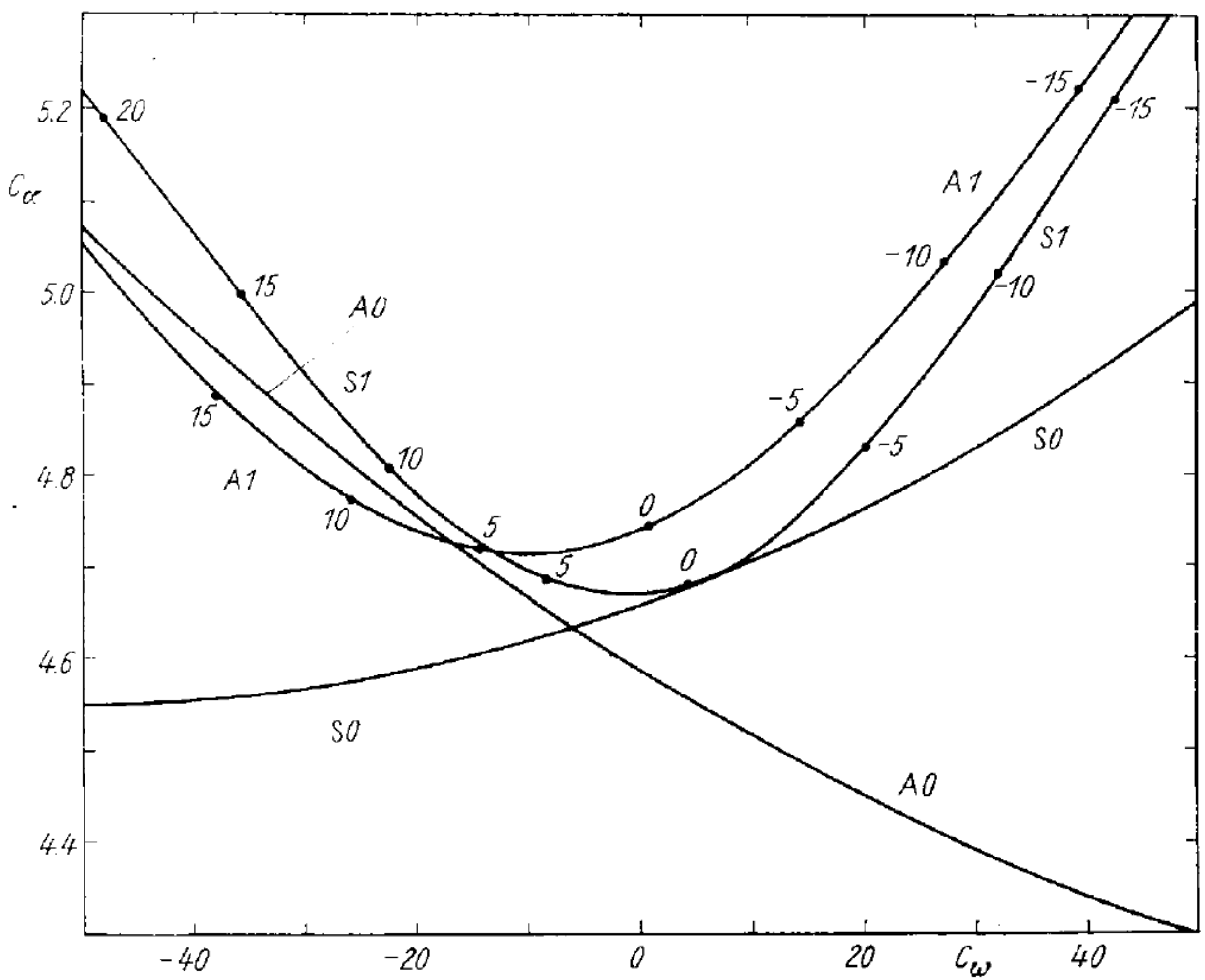



Fig. 16.11. Model including $\alpha_1$ and $\omega$-effect; $\tilde{\alpha}_1$ given by (16.4a) and (16.6); $\chi$-profile of type $a$, $x_{\chi1} + d_{\chi1} \leq 0$, $x_{\chi2} = 0.5$, $d_{\chi2} = 0.4$; $w$ given by (16.7), $x_w = 0.5$, $d_w = 0.4$. Marginal values of $C_\alpha$ in dependence on $C_\omega$. As for $\Omega$ the explanations given with figure 16.7 apply

Krause &Raedler 1980.

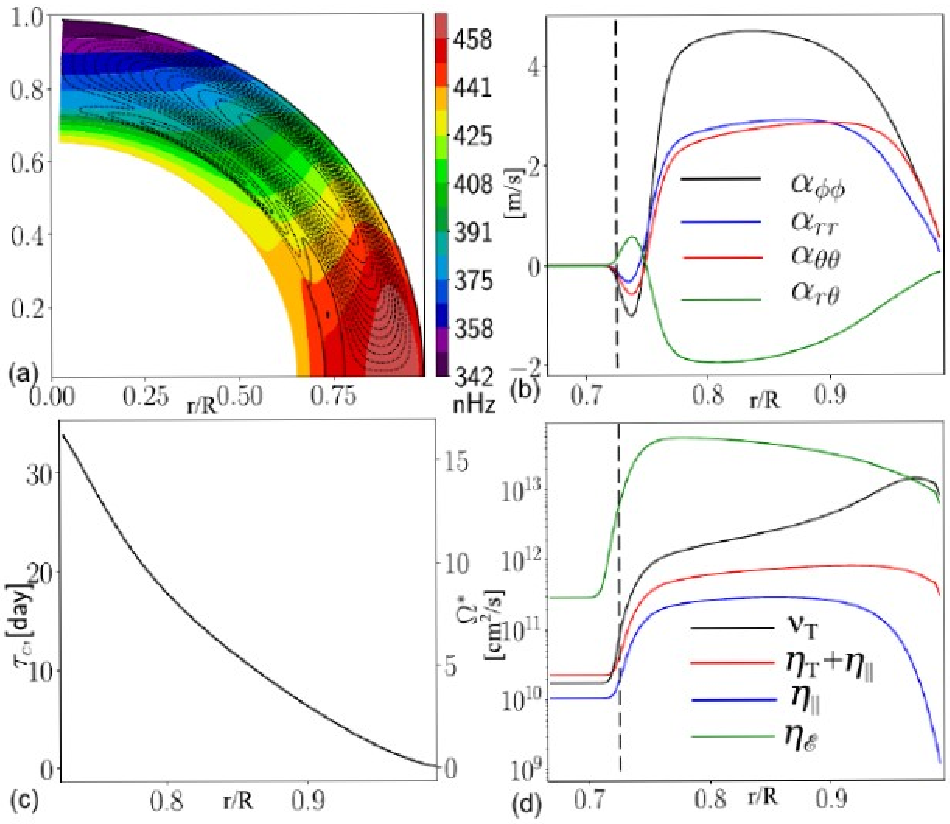
1)A0,S0 have a lower threshold than A1,S1
2) Increase $C_\omega$ results in increase the instability threshold for A1,S1
3)For the Sun, $C_\omega >> C_\alpha$

# Axisymmetric shperical dynamo

$$\frac{\partial B}{\partial t} = \frac{1}{r}\frac{\partial(\Omega, A)}{\partial(r,\theta)} + \frac{1}{r}\left(\frac{\partial r(\mathcal{E}_\theta - \langle U_r\rangle B)}{\partial r} - \frac{\partial(\mathcal{E}_r + \langle U_\theta\rangle B)}{\partial\theta}\right),$$

$$\frac{\partial A}{\partial t} = r\sin\theta\mathcal{E}_\varphi - \frac{\langle U_\theta\rangle}{r}\frac{\partial A}{\partial\theta} - \langle U_r\rangle\frac{\partial A}{\partial r},$$

and we seek for $B, A \sim \exp(\lambda t)$



# Axisymmetric dynamo, dynamo instability analysis

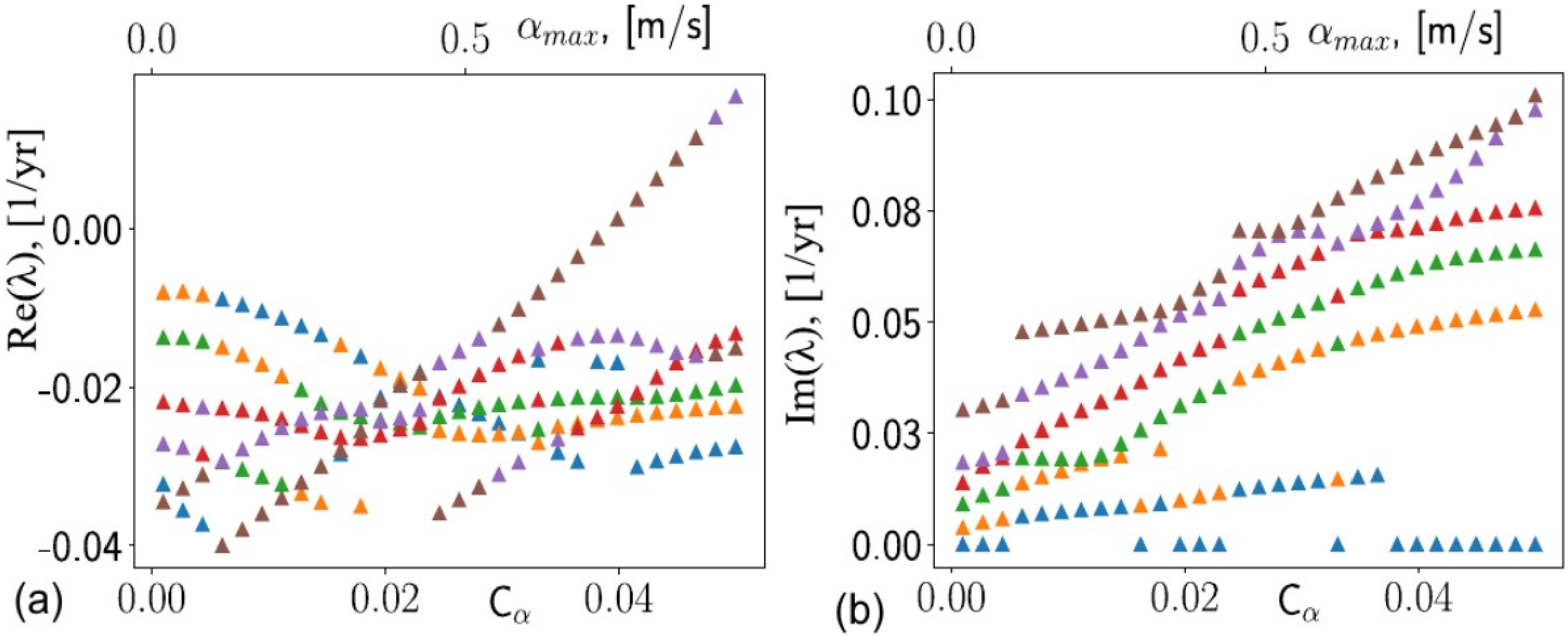
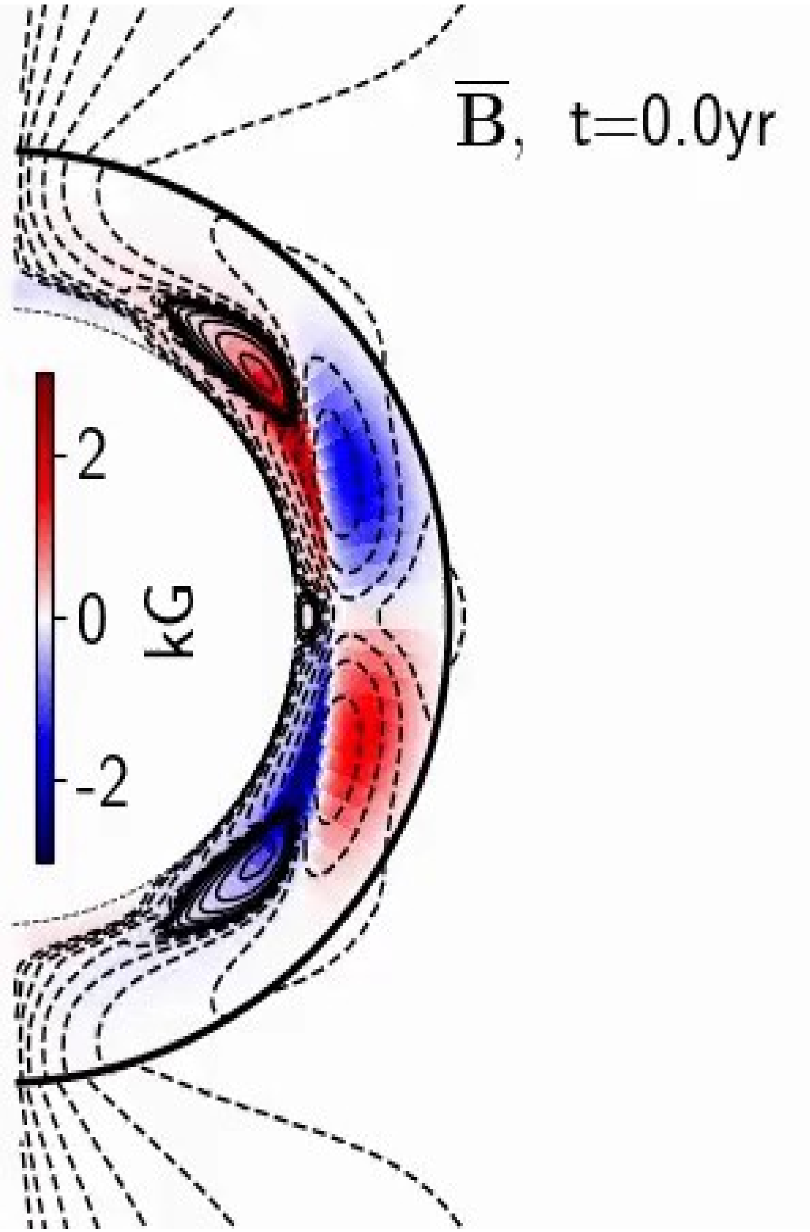
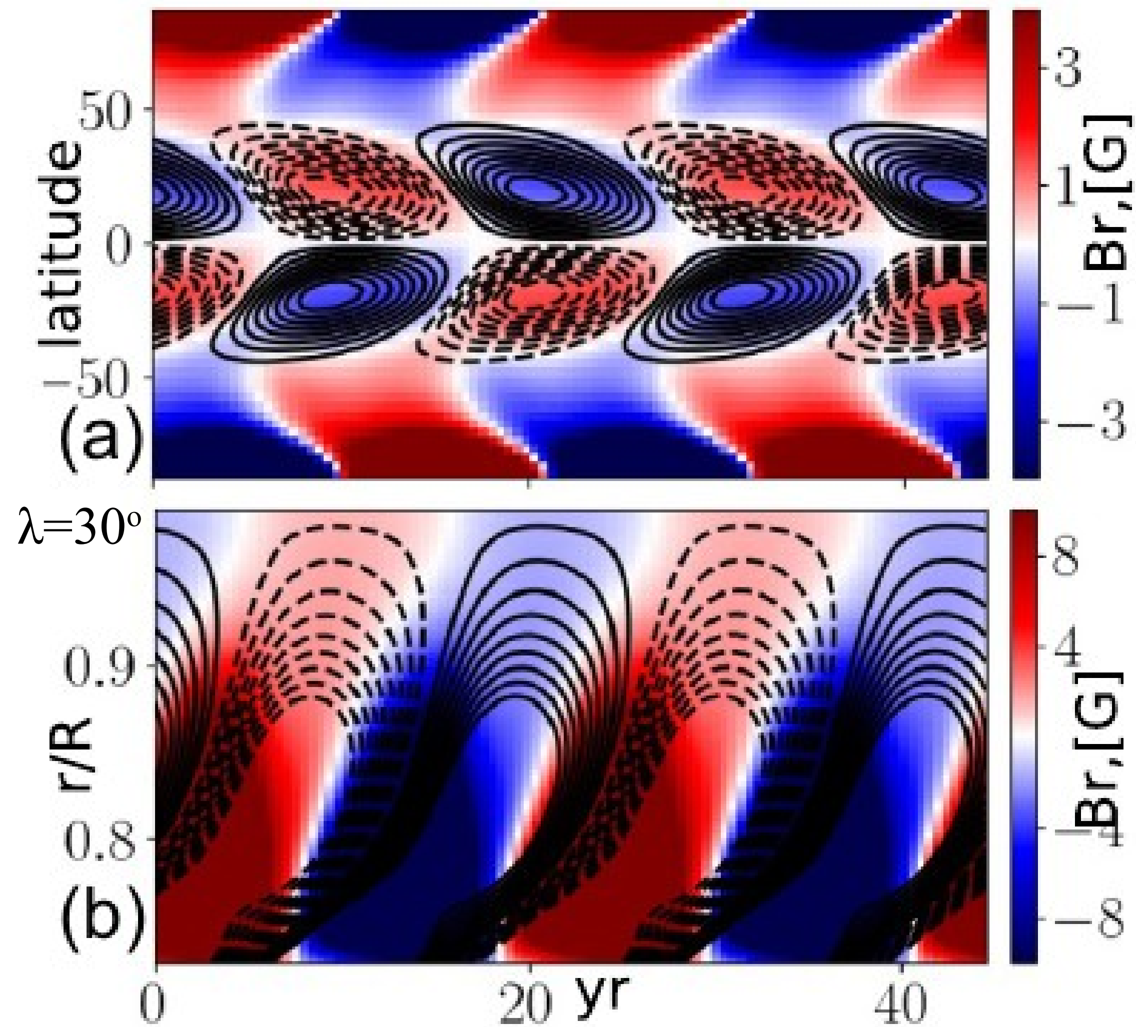



**Figure 2.** (a) Growth rates of the first six eigen odd dynamo modes for the solar type dynamo model with the local mean electromotive force, the *x*-axis show the maximum magnitude of the $\alpha_{\phi\phi}$ component in the convection zone; colours mark the different eigen modes; (b) shows the eigen frequency for each dynamo

# Solar dynamo waves, axisymmetric model



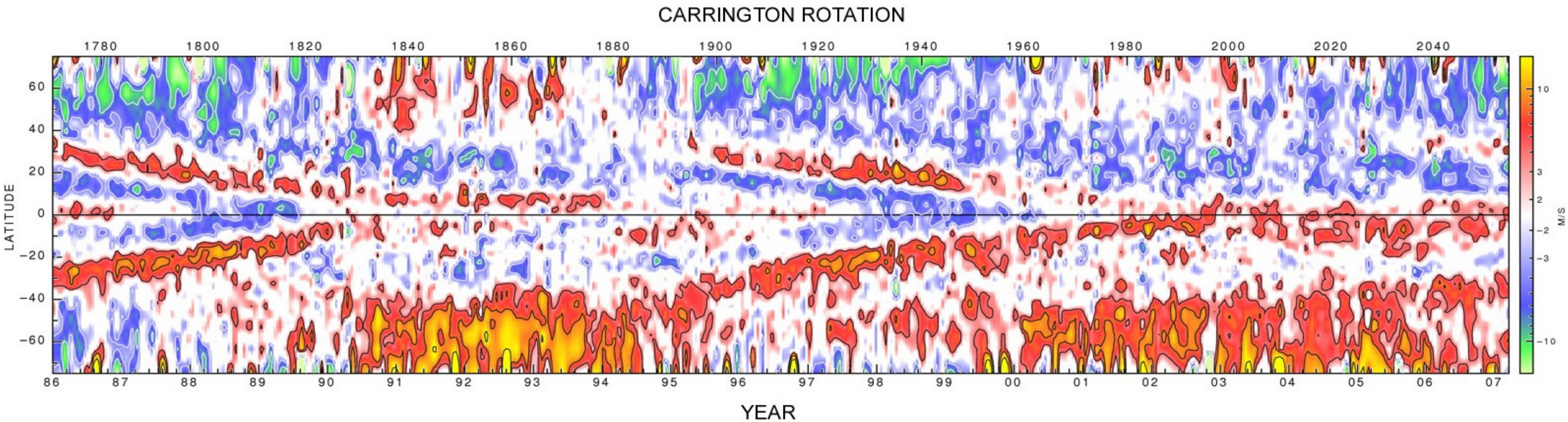




$B_T/B_P \sim 150$, $P_{dyn}=21yr$

# Torsional oscillations and “extended cycle”

- Solar zonal oscillations of angular velocity as $B^2$ effect of MA on the angular momentum transport.
- Though, the mechanical action of the large and small-scale Lorentz forces modulated by solar cycle being the essential part of the play, it does not explain the extended mode of the torsional oscillations.
- “Extended cycle” shows itself in many others parameters of SA

**PHOTOSPHERIC VELOCITY FIELDS**




Courtesy of Roger Ulrich (http://obs.astro.ucla.edu/torsional.html)

# Overview from SOHO and SDO

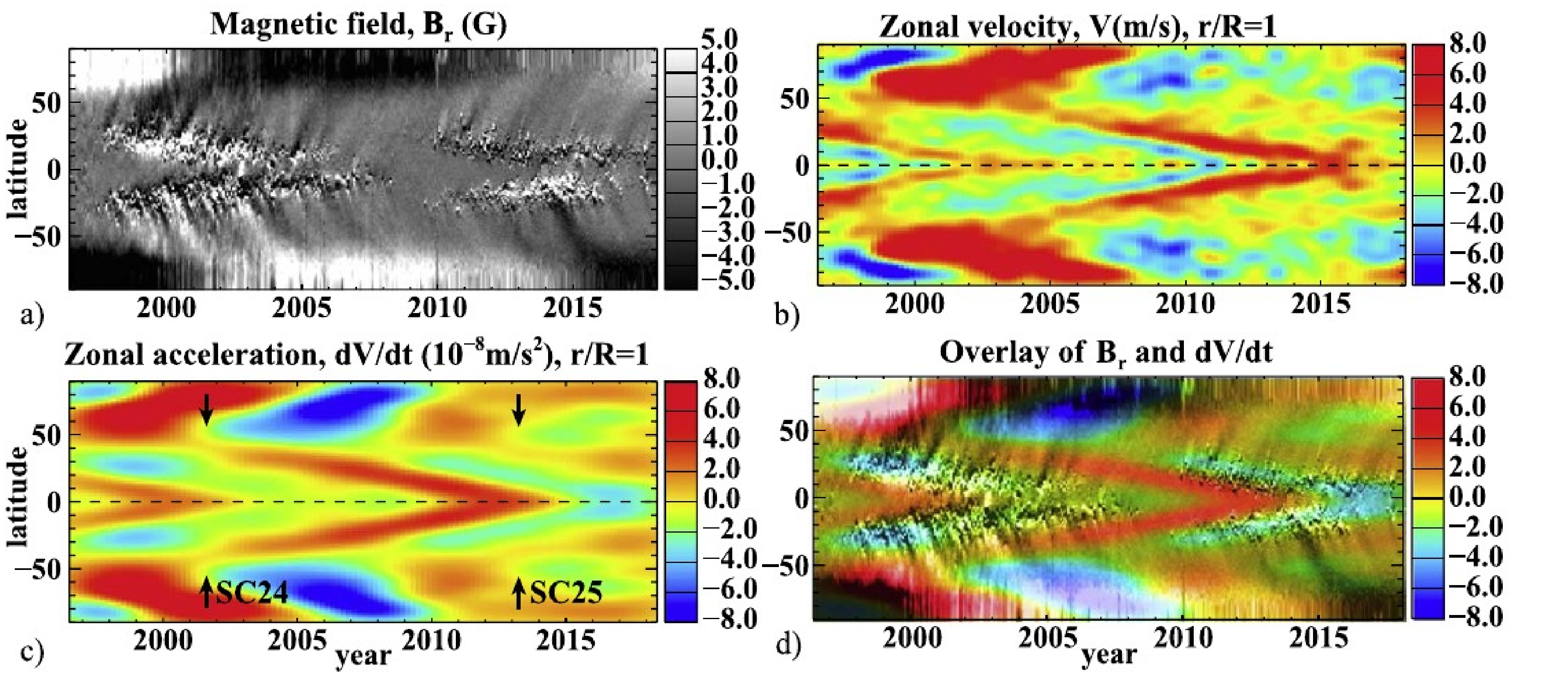



**Figure 1.** Observational results from *Solar and Heliospheric Observatory*/MDI and *Solar Dynamics Observatory* (*SDO*)/HMI. Shown are the time–latitude diagrams of (a) evolution of radial magnetic field during the last two solar cycles; (b) zonal flow velocity ("torsional oscillations"); (c) zonal acceleration calculated after applying a Gaussian spatial and temporal filter, where arrows indicate the start of extended Solar Cycles 24 and 25 at about 55° latitude, defined as a starting point of the zonal deceleration (blue areas); and (d) overlay of the zonal acceleration (color image) and the radial magnetic field (gray scale) (after Kosovichev & Pipin 2019).

# Conservation of angular momentum

$$\frac{\partial}{\partial \mathrm{t}}\overline{\rho}\mathrm{r}^2\sin^2\theta\Omega = -\nabla\cdot\left(r\sin\theta\left(\overline{\rho}\hat{\mathbf{T}}_\phi + r\overline{\rho}\sin\theta\Omega\overline{\mathbf{U}}^{\mathbf{m}}\right)\right)$$

$$+ \quad \nabla\cdot\left(r\sin\theta\frac{\langle\mathbf{B}\rangle\langle B_\phi\rangle}{4\pi}\right)$$

$$\hat{\mathrm{T}}_{\mathrm{ij}} = \left(\langle u_i u_j\rangle - \frac{1}{4\pi\overline{\rho}}\left(\langle b_i b_j\rangle - \frac{1}{2}\delta_{ij}\langle\mathbf{b}^2\rangle\right)\right)$$

$$= \delta_{ij}p_T + \Lambda_{ijk}\Omega_k - \mathcal{N}_{ijkl}\frac{\partial\overline{U}_k}{\partial x_l} + \ldots$$

For a turbulent flow under rotation and magnetic field:

$$\mathcal{N}_{ijkl} \propto f(\delta_{ij}, \delta_{kl}, \Omega_i\Omega_j, B_iB_j, \ldots)$$

$$\Lambda_{ijk} \propto f(\varepsilon_{ijk}, \Omega^*, |B|, \ldots)$$

(Kitchatinov et al 1993, Kueker et al 1996, Kitchatinov& Ruediger 2005)

# Simple model

Consider the spherical coordinate system: $r, \theta, \varphi$; axisymmetric rotational flow: $(0, 0, \mathrm{rsin}\theta\Omega(r, \theta))$; the toroidal field is winding from poloidal by DR (neglect turbulent effects):

$$\frac{\partial B_\varphi}{\partial t} = (\mathbf{B}^{(p)} \cdot \nabla)(\mathrm{rsin}\theta\Omega), \quad (1)$$
$$\frac{\partial \mathbf{B}^{(p)}}{\partial t} = 0$$

Consider a perturbation of the azimuthal component of equation of motion (previous slide) and let us $\boldsymbol{B}^p$ is uniform

$$\bar{\rho} r^2 \sin^2\theta \frac{\partial^2 \delta\Omega}{\partial t^2} = (\mathbf{B}^{(p)} \cdot \nabla)(\mathrm{rsin}\theta B_\varphi)$$
$$= (\mathbf{B}^{(p)} \cdot \nabla)^2 (r^2 \sin^2\theta\Omega)$$

$$\bar{\rho} r^2 \sin^2\theta \frac{\partial^2 \delta\Omega}{\partial t^2} = (\mathbf{B}^{(p)} \cdot \nabla)^2 (r^2 \sin^2\theta \Omega)$$

If $\boldsymbol{B}^{(p)}$ is cyclic, e.g., the solar magnetc cycle than the frequency of $\delta\Omega$ is twice as compared to the cycle of $\boldsymbol{B}^{(p)}$

Amplitude of acceleration:

$$\frac{\partial r \sin\theta \delta\Omega}{\partial t} = \frac{|\mathbf{B}^{(p)}|^2}{2\pi\bar{\rho}} \Omega \sin\theta$$

For $|\mathbf{B}_p| \sim 1$G, $\Omega = 2.86 \cdot 10^{-6}$Hz, $\bar{\rho} \sim 0.1 g/\mathrm{cm}^3$ (bottom of the SCZ) we estimate the acceleration to be $\sim 10^{-7} m/s^2$ . This is the minimum! because for the top of the convection zone we would have $\sim 10^{-5} m/s^2$!

# Meridional circulation

$$\boldsymbol{\nabla}\cdot\overline{\rho}\overline{\mathbf{U}} = 0, \qquad \omega = \left(\boldsymbol{\nabla}\times\overline{\mathbf{U}}^{m}\right)_{\phi}$$

Dissipation & advection

$$\frac{\partial\omega}{\partial \mathrm{t}} = \mathrm{r}\sin\theta\,\boldsymbol{\nabla}\cdot\left(\frac{\hat{\boldsymbol{\phi}}\times\boldsymbol{\nabla}\cdot\overline{\rho}\hat{\mathbf{T}}}{\mathrm{r}\overline{\rho}\sin\theta} - \frac{\overline{\mathbf{U}}^{\mathrm{m}}\overline{\omega}}{\mathrm{r}\sin\theta}\right)$$

$$+ \quad \mathrm{r}\sin\theta\frac{\partial\Omega^2}{\partial z} - \frac{\mathrm{g}}{\mathrm{c_p r}}\frac{\partial\overline{\mathrm{s}}}{\partial\theta}$$

Generation by imbalance of centrifugal and baroclinic forces

$$+ \quad \frac{1}{4\pi\overline{\rho}}\left(\langle\mathbf{B}\rangle\cdot\boldsymbol{\nabla}\right)\left(\boldsymbol{\nabla}\times\langle\mathbf{B}\rangle\right)_{\phi} - \frac{1}{4\pi\overline{\rho}}\left(\left(\boldsymbol{\nabla}\times\langle\mathbf{B}\rangle\right)\cdot\boldsymbol{\nabla}\right)\langle\mathbf{B}\rangle_{\phi},$$

Quenching

# Heat transport

$$\overline{\rho}\overline{T}\left(\frac{\partial\overline{\mathrm{s}}}{\partial t}+\left(\overline{\mathbf{U}}\cdot\boldsymbol{\nabla}\right)\overline{\mathrm{s}}\right)=-\boldsymbol{\nabla}\cdot\left(\mathbf{F}^{c}+\mathbf{F}^{r}\right)-\hat{T}_{ij}\frac{\partial\overline{U}_{i}}{\partial r_{j}}-\boldsymbol{\mathcal{E}}\cdot\left(\nabla\times\overline{\boldsymbol{B}}\right),$$

$$\mathrm{F_i^{conv}} = -\mathrm{c_p}\overline{\rho}\overline{\mathrm{T}}\kappa_{\mathrm{ij}}\nabla_{\mathrm{j}}\overline{\mathrm{s}},$$

(Kitchatinov et al 1993, Pipin 2004)

$$\kappa_{\mathrm{ij}} = \kappa_{\mathrm{T}}\left(\phi_{\kappa}^{(\mathrm{I})}\left(\beta\right)\phi\left(\Omega^{*}\right)\delta_{\mathrm{ij}} + \phi_{\kappa}^{(\mathrm{A})}\left(\beta\right)\phi_{\parallel}\left(\Omega^{*}\right)\frac{\Omega_{\mathrm{i}}\Omega_{\mathrm{j}}}{\Omega^{2}}\right).$$

Mixing length approximation:

$$\mathrm{u}' = \frac{\ell}{2}\sqrt{-\frac{\mathrm{g}}{\mathrm{c_p}}\frac{\partial\overline{\mathrm{s}}}{\partial\mathrm{r}}}.$$

Reference state of g, ρ, T, $c_p$ etc. from MESA solar interior model

# Boundary conditions

Stress-free at the surface and solid body rotation below overshoot region (see, Pipin&Kosovichev 2020)

Given heat flux at bottom & black-body radiation at the top

Zero B at the bottom of overshoot

Either vacuum BC, or more “realistic” BC with penetration of toroidal B to the surface and the potential or pure poloidal field outside (Moss et al 1992)

# Driving forces

# Conservation of angular momentum

$$\frac{\partial}{\partial \mathrm{t}}\overline{\rho}\mathrm{r}^2\sin^2\theta\Omega \;=\; -\boldsymbol{\nabla}\cdot\left(r\sin\theta\left(\overline{\rho}\hat{\mathbf{T}}_\phi + r\overline{\rho}\sin\theta\Omega\overline{\mathbf{U}}^{\mathbf{m}}\right)\right)$$

$$+\quad \boldsymbol{\nabla}\cdot\left(r\sin\theta\frac{\langle\mathbf{B}\rangle\langle B_\phi\rangle}{4\pi}\right)$$

$$\hat{\mathrm{T}}_{\mathrm{ij}} \;=\; \left(\langle u_i u_j\rangle - \frac{1}{4\pi\overline{\rho}}\left(\langle b_i b_j\rangle - \frac{1}{2}\delta_{ij}\langle\mathbf{b}^2\rangle\right)\right)$$

$$=\; \delta_{ij}p_T + \Lambda_{ijk}\Omega_k - \mathcal{N}_{ijkl}\frac{\partial\overline{U}_k}{\partial x_l} + \ldots$$

For a turbulent flow under rotation and magnetic field:

$$\mathcal{N}_{ijkl} \propto f(\delta_{ij}, \delta_{kl}, \Omega_i\Omega_j, B_iB_j, \ldots)$$

$$\Lambda_{ijk} \propto f(\varepsilon_{ijk}, \Omega^*, |B|, \ldots)$$

(Kitchatinov et al 1993, Kueker et al 1996, Kitchatinov& Ruediger 2005)

# Driving forces

$$F_I = -\frac{1}{\overline{\rho} r \sin\theta} \boldsymbol{\nabla}\cdot \left( r \sin\theta \overline{\rho} \hat{\mathbf{T}}_{\phi} \left( \boldsymbol{B} = 0 \right) \right) \quad (1)$$

Hydrodinamic inertia force

$$F_{\ell} = -\frac{1}{\overline{\rho} r \sin\theta} \boldsymbol{\nabla}\cdot \left( r \sin\theta \overline{\rho} \left\{ \hat{\mathbf{T}}_{\phi} - \hat{\mathbf{T}}_{\phi} \left( \boldsymbol{B} = 0 \right) \right\} \right), \quad (2)$$

Lorentz force caused by small-scale perturbations of mean field

$$F_L^{(t)} = \frac{1}{\overline{\rho} r \sin\theta} \boldsymbol{\nabla}\cdot \left( r \sin\theta \frac{\overline{\mathbf{B}} \overline{B}_{\phi}}{4\pi} \right), \quad (3)$$

Toroidal Lorentz force of the LS MF

$$\begin{aligned} F_L^{(p)} &= \frac{1}{4\pi\overline{\rho}} \left( \overline{\mathbf{B}} \cdot \boldsymbol{\nabla} \right) \left( \boldsymbol{\nabla} \times \overline{\mathbf{B}} \right)_{\phi} - \frac{1}{4\pi\overline{\rho}} \left( \left( \boldsymbol{\nabla} \times \overline{\mathbf{B}} \right) \cdot \boldsymbol{\nabla} \right) \overline{\mathbf{B}}_{\phi} \quad (4) \\ &+ \frac{1}{\overline{\rho}^2} \left[ \boldsymbol{\nabla} \overline{\rho} \times \left( \boldsymbol{\nabla} \frac{\overline{\mathbf{B}}^2}{8\pi} - \frac{\left( \overline{\mathbf{B}} \cdot \boldsymbol{\nabla} \right) \overline{\mathbf{B}}}{4\pi} \right) \right]_{\phi} \end{aligned}$$

Poloidal Lorentz force

$$F_U = -\frac{1}{\overline{\rho} r \sin\theta} \boldsymbol{\nabla}\cdot \left( r^2 \sin^2\theta \overline{\rho} \Omega \overline{\mathbf{U}}^{\mathbf{m}} \right), \quad (5)$$

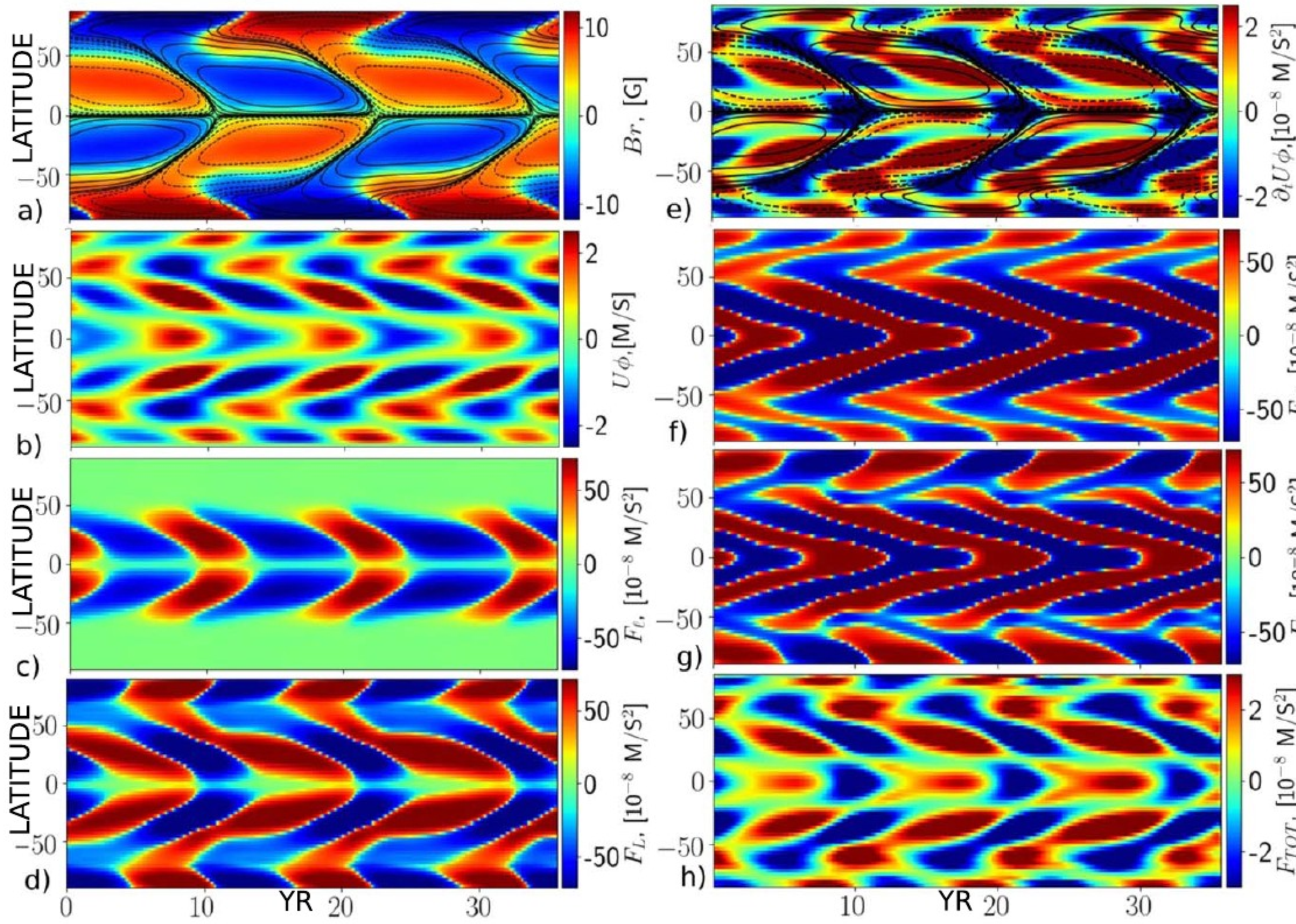
Toroidal force form meridional circulation

$$F_H = -\frac{1}{\overline{\rho} r \sin\theta} \boldsymbol{\nabla}\cdot \left( r \sin\theta \overline{\rho} \hat{\mathbf{T}}_{\phi}^{(\Lambda)} \left( H^{(0)} \right) \right), \quad (6)$$

Toroidal force from magnetic $\Lambda$ effect

See details in Pipin & Kosovichev 2019

# Dynamo induced forces




**The torsional oscillations results from a balance of different forces. Acceleration from each of them exceed the balance more than order of magnitude!**

# Conditions for the extended mode of the torsional oscilltions

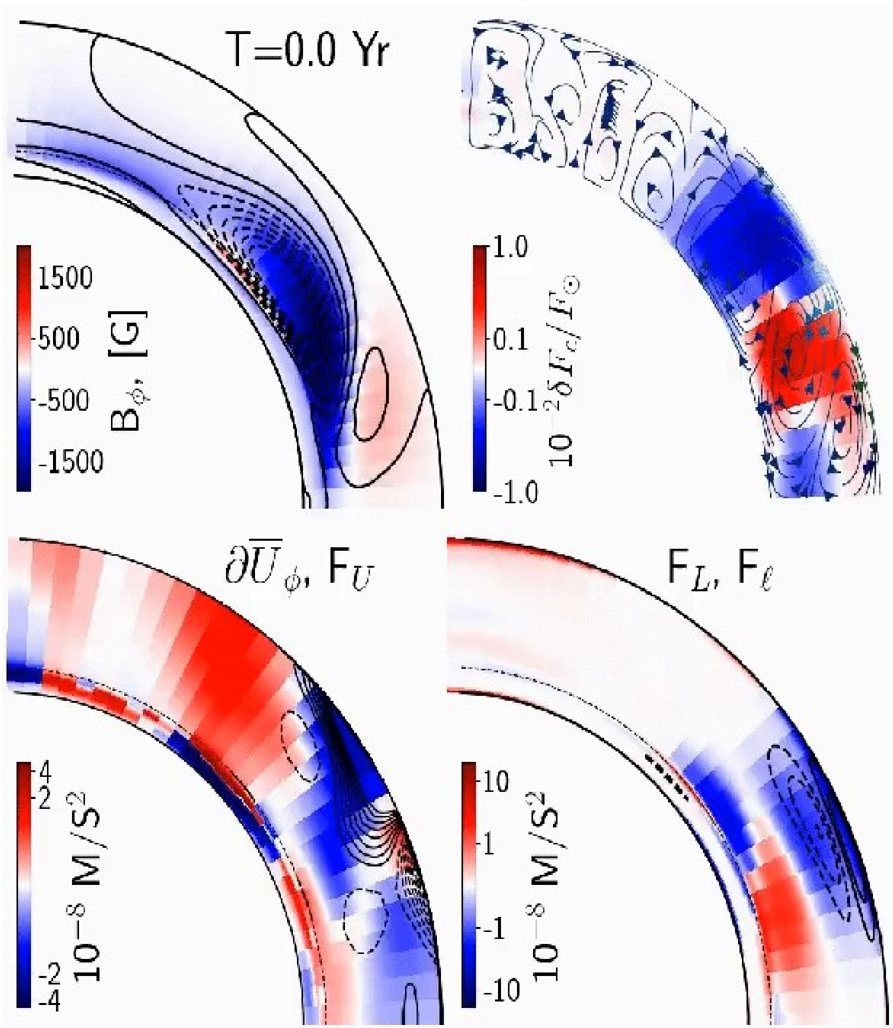
**We find that extended mode of the torsional oscillation is induced if: there are**

**1) the magnetic cycle overlap**

**2) dynamo induced perturbations of the convective heat flux**

**3)effect of MC on the TO is taken into account**

# Torsional oscillations in convection zone

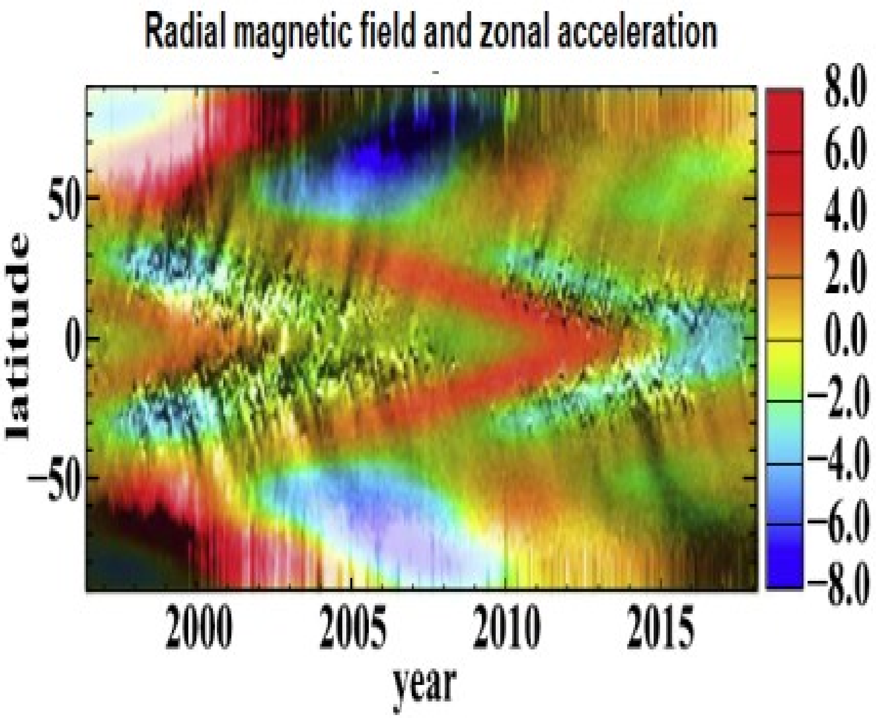


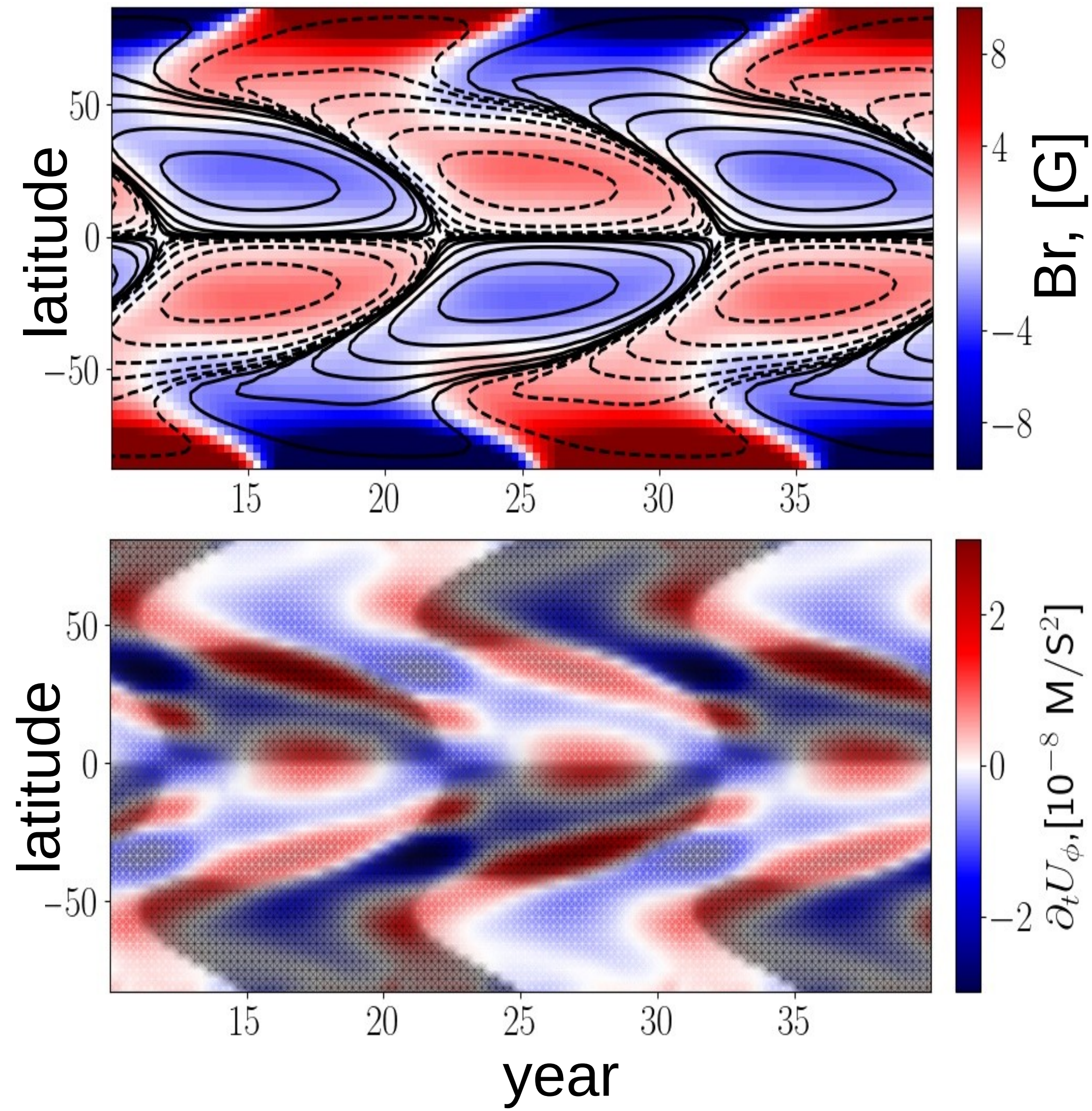

1) $B^2$ effect of the dynamo on the heat transport produces the track of 4-5 meridional circulation perturbation cells along latitude

2)The mechanical action from quenching of turbulent stresses and the mean Lorentz force cause the zonal variations of rotation.

3) The track of meridional circulation perturbation cells transports zonal variations from pole to equator

# Comparison with observations




**Extended mode TO is induced if: there are**
**1) the magnetic cycle overlap**
**2) dynamo induced perturbations of the convective heat flux**
**3)effect of MC on the TO is taken into account**

# Summary

- The mean-field evolution equations can be derived from  the standard MHD equations. The estimations of turbulent effects for the case of high Re, Rm is not a problem, as well.
- Depending on the task being solved we can choose a simplification of the dynamo model, e.g., 0D, 1D, 2D, 3D models can be  applied
- The  dynamo is a large-scale instability which require the critical threshold
- The 11-th and 20 years torsional oscillations of the Sun results from the nonlinear balance of the dynamo induced azimuthal forces in the presence of the magnetic perturbation of the heat transport inside the convection zone

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